Differentiable Manifolds

§21. Orientations

Sichuan University, Fall 2020
Orientations of a Vector Space

Example (Orientations of \( \mathbb{R} \))

On \( \mathbb{R} \) an orientation is one of two directions:

\[
\begin{array}{c}
\rightarrow \\
\leftarrow \\
\end{array}
\]

The orientations of a line.

Two (nonzero) vectors \( u \) and \( v \) define the same direction if and only if \( u = av \) with \( a > 0 \).
On $\mathbb{R}^2$ an orientation is either direct (counterclockwise) or indirect (clockwise).

![The orientations of a plane.](image)

- An ordered basis $(v_1, v_2)$ defines the direct (resp., indirect) orientation if the angle $\theta$ from $v_1$ to $v_2$ is $> 0$ (resp., $< 0$).
- As $\det(v_1, u_2) = |v_1||v_2|\sin \theta$, we see that

$$(v_1, v_2) \text{ is direct } \iff \det(v_1, v_2) > 0,$$

$$(v_1, v_2) \text{ is indirect } \iff \det(v_1, v_2) < 0.$$
Orientations of a Vector Space

Example (The orientations of a plane, continued)

- Let \((u_1, u_2)\) and \((v_1, v_2)\) be ordered bases. Write \(u_i = \sum a^j_i v_j\).
  - The matrix \(A = [a^j_i]\) is called the change-of-basis matrix. We have
    \[
    \det(u_1, u_2) = \det(A) \det(v_1, v_2).
    \]
- Thus, \((u_1, u_2)\) and \((v_1, v_2)\) defines the same orientation if and only if \(\det(A) > 0\).

Definition

Two bases \((u_1, u_2)\) and \((v_1, v_2)\) are called equivalent if the change-of-basis matrix has positive determinant.

- This defines an equivalence relation on order bases.
- We have a one-to-one correspondance:
  \[
  \{\text{orientations}\} \leftrightarrow \{\text{equivalence classes of bases}\}.
  \]
Orientations of a Vector Space

**Definition**

Let $V$ be a vector space of dimension $n$. Two bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are said to be *equivalent*, and we write $(u_1, \ldots, u_n) \sim (v_1, \ldots, v_n)$ if we can go from one to the other by a change-of-base matrix with positive determinant.

**Remark**

This defines an equivalence relation on bases of $V$.

**Definition**

An *orientation* of $V$ is a choice of an equivalence class of bases.

**Remark**

- A vector space has exactly two orientations.
- We denote by $[(v_1, \ldots, v_n)]$ the class of $(v_1, \ldots, v_n)$. 
Orientations and Covectors

Remark

Let \((v_1, \ldots, v_n)\) be a basis of a vector space \(V\). Let \((\alpha^1, \ldots, \alpha^n)\) be the dual basis of \(V^*\). Then, for any \(n\)-covector \(\beta \in \Lambda^n(V^*)\), we have

\[
\beta = \beta(v_1, \ldots, v_n)\alpha^1 \wedge \cdots \wedge \alpha^n.
\]

In particular, \(\beta \neq 0\) if and only if \(\beta(v_1, \ldots, v_n) \neq 0\).
Lemma (Lemma 21.1)

Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be vectors in $V$ such that $u_i = \sum a^i_j v_j$ for some matrix $A = [a^i_j]$. For any $n$-covector $\beta$ we have

$$\beta(u_1, \ldots, u_n) = (\det A)\beta(v_1, \ldots, v_n).$$

Consequence

Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be bases and $\beta \neq 0$. Then $\beta(u_1, \ldots, u_n)$ and $\beta(v_1, \ldots, v_n)$ have same sign if and only if $\det A > 0$, i.e., $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ define the same orientation.
**Definition**

We say that an $n$-covector $\beta$ on $V$ **specifies** the orientation $[(v_1, \ldots, v_n)]$ if $\beta(v_1, \ldots, v_n) > 0$.

**Remark**

Let $(v_1, \ldots, v_n)$ be a basis of a vector space $V$. Let $(\alpha_1, \ldots, \alpha_n)$ be the dual basis of $V^*$. By the remark on slide 6, we have

$$\beta = \beta(v_1, \ldots, v_2)\alpha_1 \wedge \cdots \wedge \alpha_n$$

Thus, $\beta$ **specifies** the orientation $[(v_1, \ldots, v_n)]$ if and only if $\beta$ is a positive scalar multiple of $\alpha_1 \wedge \cdots \wedge \alpha_n$. 
Definition
We say that two non-zero \( n \)-covectors \( \beta \) and \( \beta' \) are equivalent if \( \beta' = a \beta \) with \( a > 0 \).

Remark
This defines an equivalence relation on \( \Lambda^n(V^*) \setminus \{0\} \).

Fact
We have a one-to-one correspondence:
\[
\{ \text{orientations of } V \} \leftrightarrow \{ \text{equivalence classes of } n \text{-covectors } \neq 0 \}.
\]
Fact

Let $M$ be a smooth manifold of dimension $n$. If $(X_1, \ldots, X_n)$ is a frame of $TM$ over $U$ and $p \in U$, then $(X_{1,p}, \ldots, X_{n,p})$ is a basis of $T_pM$, and hence it defines an orientation of $T_pM$.

Remark

We say that a frame $(X_1, \ldots, X_n)$ of $TM$ over an open $U$ is continuous, if, for each $i$, the vector field $X_i$ is continuous as a map from $U$ to $TM$. 
Definition (Pointwise orientation)

- A pointwise orientation of $M$ assigns to each $p \in M$ an orientation of $T_p M$, i.e., an equivalence class $\mu_p = [(X_{1,p}, \ldots, X_{n,p})]$ of (ordered) bases of $T_p M$.

- We say that a pointwise orientation is continuous at $p \in M$ if there is an open $U$ containing $p$ and a continuous tangent frame $(Y_1, \ldots, Y_n)$ over $U$ such that $(Y_{1,q}, \ldots, Y_{n,q})$ defines the orientation of $T_q M$ for every $q \in U$. 

Orientations of a Manifold
Orientations of a Manifold

Definition (Orientations)

- An orientation of $M$ is a pointwise orientation which is continuous at every $p \in M$.
- We say that $M$ is orientable when it admits an orientation.
- We say that $M$ is oriented when it is equipped with an orientation.

Remarks

- Any continuous (or even smooth) global frame $(X_1, \ldots, X_n)$ of $TM$ over $M$ defines an orientation.
- The converse does not hold. For instance, the even-dimensional spheres $S^{2n}$, $n \geq 1$, do not admit global tangent frames; yet there are orientable.
Orientations of a Manifold

Example

$\mathbb{R}^n$ is oriented by the global frame $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$. More generally, any vector space is orientable.

Example (see also Problem 21.7)

If $G$ is a Lie group, then $G$ admits a global tangent frame consisting of left-invariant vector fields, and so $G$ is orientable.
Example (Möbius Band; Example 21.2)

The Möbius band is the quotient of the rectangle \( R = [0, 1] \times [-1, 1] \) by the equivalence relation,

\[
(x, y) \sim (x, y), \quad 0 < x < 1, \quad -1 \leq y \leq 1,
\]
\[
(0, y) \sim (1, -y), \quad -1 \leq y \leq 1.
\]

This is a non-orientable surface (see Tu’s book).
Proposition (Proposition 21.3)

If an orientable manifold is connected, then it has exactly two possible orientations.
Lemma (see Lemma 21.4)

Let $\mu$ be a pointwise orientation of $M$. TFAE:

(i) $\mu$ is continuous on $M$.

(ii) For every $p \in M$, there is a chart $(U, x^1, \ldots, x^n)$ near $p$ such that the orientation of $T_p M$ is defined by $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$.

(iii) For every $p \in M$, there is a chart $(U, x^1, \ldots, x^n)$ near $p$ such that the orientation of $T_p M$ is specified by $dx^1 \wedge \cdots \wedge dx^n$.

Theorem (Theorem 21.5)

A manifold $M$ of dimension $n$ is orientable if and only if there exists a smooth nowhere-vanishing $n$-form on $M$.
Remark

Let $\omega$ be a nowhere vanishing $n$-form on $M$. Then $\omega$ defines an orientation of $M$ as follows:

- For every $p \in M$, there is a chart $(U, x^1, \ldots, x^n)$ near $p$ such that $\omega(\partial/\partial x^1, \ldots, \partial/\partial x^n) > 0$ on $U$.
- The orientation of $T_p M$ is the class of $(\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p)$.
- As the frames $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$ are continuous (since they are smooth), we get a continuous pointwise orientation on $M$, i.e., an orientation of $M$. 

Example

Suppose that 0 is a regular value of some smooth function \( f(x, y, z) \) on \( \mathbb{R}^3 \).

- By the regular level set theorem, the zero set \( S = f^{-1}(0) \) is a regular submanifold of \( \mathbb{R}^3 \), and hence is manifold.
- By Problem 19.11 it admits a smooth nowhere-vanishing 2-form.
- Thus, by Theorem 21.5 the manifold \( S \) is orientable.

For instance, the 2-sphere \( S^2 \) is orientable.
Definition

We say that two $C^\infty$ nowhere-vanishing $n$-forms $\omega$ and $\omega'$ on $M$ are equivalent, and we write $\omega \sim \omega'$, if there is $f \in C^\infty(M)$, $f > 0$, such that $\omega' = f \omega$.

Remark

This defines an equivalence relation on $C^\infty$ nowhere-vanishing $n$-forms on $M$.

Proposition

We have a one-to-one correspondence:

\[
\{\text{orientations of } M\} \leftrightarrow \left\{ C^\infty \text{ equivalence classes of nowhere-vanishing } n\text{-forms} \right\}
\]
Definition
If $\omega$ is a $C^\infty$ nowhere-vanishing $n$-form that specifies the orientation at every point, then we say that $\omega$ is an orientation form.

Example
The (standard) orientation of $\mathbb{R}^n$ is specified by the $n$-form $dx^1 \wedge \cdots \wedge dx^n$.

Remark
An oriented manifold is often represented as $(M, [\omega])$, where $[\omega]$ is a class of orientation forms.
Definition

A diffeomorphism $F : (N, [\omega_N]) \rightarrow (M, [\omega_M])$ between oriented manifolds is called orientation-preserving if $[F^*\omega_M] = [\omega_N]$. It is called orientation-reversing if $[F^*\omega_M] = [-\omega_N]$.

Proposition (Proposition 21.8)

Let $U$ and $V$ be open sets in $\mathbb{R}^n$ equipped with orientations inherited from $\mathbb{R}^n$. A diffeomorphism $F : U \rightarrow V$ is orientation-preserving if and only if the Jacobian determinant $\det[\partial F^i / \partial x^j]$ is everywhere positive on $U$. 
**Definition (Definition 21.9)**

An atlas of $M$ is called *oriented* if given two overlapping charts $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ the transition map is orientation-preserving, i.e., the Jacobian determinant $\det[\partial y^i/\partial x^j]$ is everywhere positive on $U \cap V$.

**Theorem (Theorem 21.10)**

*A manifold $M$ is orientable if and only if it admits an oriented atlas.*
Remark

An oriented atlas defines an orientation of $M$ as follows:

- Given $p \in M$ and a chart $(U, x^1, \ldots, x^n)$, the orientation of $T_p M$ is the class of $(\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p)$.
- The orientation of $T_p M$ does not depend on the choice of the chart, since the atlas is oriented.
- As the frames $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$ are continuous, we get a continuous pointwise orientation on $M$, i.e., an orientation of $M$. 
Definition (Definition 21.11)

Two oriented atlases \( \{(U_\alpha, \phi_\alpha)\} \) and \( \{(V_\beta, \psi_\beta)\} \) on \( M \) are said to be equivalent if the transition functions

\[
\phi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(U_\alpha \cap V_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap V_\beta)
\]

have positive Jacobian determinants for all \( \alpha, \beta \).

Remark

This defines an equivalence relation on oriented atlases.

Proposition

We have a one-to-one correspondence:

\[
\{ \text{orientations of } M \} \leftrightarrow \{ \text{equivalence classes of oriented atlases} \}
\]
Summary
If $M$ is an orientable manifold of dimension $n$, there are 3 equivalent ways to define an orientation:

1. By using a continuous pointwise orientation.
2. By using a smooth nowhere-vanishing $n$-form.