Differentiable Manifolds

§7. Quotients

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An equivalence relation on a set $S$ is given by a subset $R \subset S \times S$ with the following properties:

- **Transitivity**: $(x, x) \in R$ for all $x \in S$.
- **Symmetry**: $(x, y) \in R \iff (y, x) \in R$.
- **Transitivity**: $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$.

When $(x, y) \in R$ we say that $x$ and $y$ are *equivalent* and write $x \sim y$.

The set $R$ is called the *graph* of the equivalence relation.
The Quotient Topology

Definition

Let \( \sim \) be an equivalence relation on \( S \).

- The *class* of \( x \in S \), denoted \([x]\), is the subset of \( S \) consisting of all \( y \in S \) that are equivalent to \( x \).
- The set of equivalence classes is denoted \( S/\sim \) and is called the *quotient* of \( S \) by \( \sim \).
- The map \( \pi : S \to S/\sim \), \( x \to [x] \) is called the *natural projection map* (or *canonical projection*)

Remarks

1. The equivalence classes form a partition of \( S \).
2. The canonical projection \( \pi : S \to S/\sim \) is always onto.
Suppose that $S$ is a topological space. Let $\mathcal{T}$ be the collection of subsets $U \subset S/\sim$ such that $\pi^{-1}(U)$ is an open in $S$.

- $\mathcal{T}$ is closed under unions and finite intersections: if $U_\alpha \in \mathcal{T}$ and $V_i \in \mathcal{T}$, then
  \[
  \pi^{-1}\left(\bigcup U_\alpha\right) = \bigcup \pi^{-1}(U_\alpha) \quad \text{and} \quad \pi^{-1}(V_1 \cap V_2) = \pi^{-1}(V_1) \cap \pi^{-1}(V_2)
  \]
  are again contained in $\mathcal{T}$.
- Therefore $\mathcal{T}$ defines a topology on $S/\sim$.

The topology $\mathcal{T}$ is called the quotient topology.

Equipped with this topology $S/\sim$ is called the quotient space of $S$ by $\sim$. 
The Quotient Topology

Remarks

1. A subset $U \subset S/\sim$ is open if and only if $\pi^{-1}(U)$ is an open in $S$.

2. This implies that the projection map $\pi : S \rightarrow S/\sim$ is automatically continuous.

3. The quotient topology is actually the strongest topology on $S/\sim$ for which the map $\pi : S \rightarrow S/\sim$ is continuous.
Continuity of a Map on a Quotient

**Fact**

Let $f : S \to Y$ be a map that is constant on each equivalence class, i.e.,

$$x \sim y \Rightarrow f(x) = f(y).$$

Then $f$ descends to a map $\bar{f} : S/\sim \to Y$ such that

$$\bar{f}([x]) = f(x), \quad x \in S.$$

**Remarks**

1. The definition of $\bar{f}$ means that if $c$ is an equivalence class in $S/\sim$, then $\bar{f}(c) = f(x)$ for any $x \in c$.

2. The equality $\bar{f}([x]) = f(x)$ for all $x \in S$ means that $\bar{f} \circ \pi = f$. That is, we have a commutative diagram,

$$
\begin{array}{ccc}
S & \xrightarrow{f} & Y \\
\pi \downarrow & & \downarrow \bar{f} \\
S/\sim & & 
\end{array}
$$
Continuity of a Map on a Quotient

Proposition (Proposition 7.1)

The induced map \( \bar{f} : S/\sim \to Y \) is continuous if and only if the original map \( f : S \to Y \) is continuous.

Corollary

A map \( g : S/\sim \to Y \) is continuous if and only if the composition \( g \circ \pi : S \to Y \) is continuous.
Fact

Let \( A \) be a subset of \( S \). We can define an equivalence relation \( \sim \) on \( S \) by declaring:

\[
\begin{align*}
  x \sim x & \quad \text{for all } x \in S, \\
  x \sim y & \quad \text{for all } x, y \in A.
\end{align*}
\]

In other words, if we let \( \Delta = \{(x, x); x \in S\} \) be the diagonal of \( S \times S \), then the graph of the relation is just

\[
\mathcal{R} = \Delta \cup (A \times A).
\]

It can be checked this is an equivalence relation.

Definition

We say that the quotient space \( S/\sim \) is obtained by identifying \( A \) to a point.
Identification of a Subset to a Point

Example

Let $I$ be the unit interval $[0, 1]$ and $I/\sim$ the quotient space by identifying $0, 1$ to a point, i.e., by identifying $0$ and $1$.

1. The equivalence classes consists of the singletons $\{t\}$, $t \in (0, 1)$, and the pair $\{0, 1\}$.

2. Let $\mathbb{S}^1 \subset \mathbb{C}$ be the unit circle, and define $f : I \rightarrow \mathbb{S}^1$ by $f(t) = e^{2i\pi t}$. As $f(0) = f(1)$ it induces a map $\bar{f} : I/\sim \rightarrow \mathbb{S}^1$.

3. The induced map $\bar{f} : I/\sim \rightarrow \mathbb{S}^1$ is continuous, since $f$ is continuous.

Proposition (Proposition 7.3)

The induced map $\bar{f} : I/\sim \rightarrow \mathbb{S}^1$ is a homeomorphism.
### A Necessary Condition for a Hausdorff Quotient

#### Facts
- If $X$ is a Hausdorff topological space, then every singleton $\{x\}$, $x \in X$, is a closed set in $X$.
- If the quotient space $S/\sim$ is Hausdorff, then every singleton $\{[x]\}$, $x \in S$, is closed in $S/\sim$. This means that the preimage $\pi^{-1}(\{[x]\}) = [x]$ is closed in $S$.

#### Proposition (Proposition 7.4)
*If the quotient space $S/\sim$ is Hausdorff, then all the equivalence classes $[x]$, $x \in S$, are closed sets in $S$.*

#### Consequence
- If there is an equivalence class that is not a closed set, then the quotient space $S/\sim$ is not Hausdorff.
Let $\sim$ be the equivalence relation on $\mathbb{R}$ obtained by identifying the open interval $(0, \infty)$ to a point. Then:

- The equivalence class $[1]$ is the whole interval $(0, \infty)$.
- As $(0, \infty)$ is not a closed set in $\mathbb{R}$, the quotient space $\mathbb{R}/\sim$ is not Hausdorff.
Reminder
A map $f : X \to Y$ is open when the image of any open set in $X$ is an open set in $Y$.

Definition
We say that an equivalence relation $\sim$ on a topological space $S$ is open when the projection $\pi : S \to S/\sim$ is an open map.

Remark
- If $A \subset S$, then $\pi(A)$ is open in $S/\sim$ if and only if $\pi^{-1}(\pi(U)) = \bigcup_{x \in A}[x]$ is an open set in $S$.
- Thus, the equivalence relation $\sim$ is open if and only if, for every open $U$ in $S$, the set $\bigcup_{x \in U}[x]$ is open in $S$. 
Example

Let $\sim$ be the equivalence relation on $\mathbb{R}$ that identifies 1 and $-1$.

- We have $[x] = \{x\}$ for $x \neq \pm 1$ and $[-1] = [1] = \{\pm 1\}$.
- For the open interval $(-2, 0)$ we get

$$\bigcup_{x \in (-2,0)} [x] = \left( \bigcup_{x \notin (-2,0)} [x] \right) \cup [-1] = (-2, 0) \cup \{1\}.$$  

- As $(-2, 0) \cup \{1\}$ is not an open set, the equivalence relation $\sim$ is not open.
Reminder

If $\sim$ is an equivalence relation, then its graph is

$$R = \{(x, y) \in S \times S; x \sim y\} \subset S \times S.$$

Theorem (Theorem 7.7)

Suppose that $\sim$ is an open equivalence relation on a topological space $S$. Then the quotient space $S/\sim$ is Hausdorff if and only if the graph $R$ of $\sim$ is closed in $S \times S$. 
Example

Let \( \sim \) be the trivial equivalence relation \( x \sim y \iff x = y \). Then:

- \([x] = \{x\} \) for all \( x \in S \).
- The graph of \( \sim \) is just the diagonal,

\[
\Delta = \{(x, x); \ x \in S\} \subset S \times S.
\]

- If \( S \) is a topological space, then the projection map \( \pi : S \rightarrow S/\sim \) is a homeomorphism.

Corollary (Corollary 7.8)

A topological space \( S \) is Hausdorff if and only if the diagonal \( \Delta \) is closed in \( S \times S \).
Proposition (Proposition 7.9)

Suppose that $\sim$ is an open equivalence relation on $S$. If $\{U_\alpha\}$ is a basis for the topology of $S$, then $\{\pi(U_\alpha)\}$ is a basis for the quotient topology on $S/\sim$.

Corollary (Corollary 7.10)

If $\sim$ is an open equivalence relation on $S$, and $S$ is second countable, then the quotient space $S/\sim$ is second countable.
Remarks

1. Intuitively speaking the real projective space $\mathbb{R}P^n$ is the set of lines in $\mathbb{R}^{n+1}$ through the origin.
2. Two non-zero vectors $x, y \in \mathbb{R}^{n+1} \setminus 0$ are the same line through the origin if and only if there is $t \neq 0$ such that $y = tx$.

Fact

1. We define an equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus 0$ by

$$x \sim y \iff y = tx \text{ for some } t \neq 0.$$ 

2. The conjugacy classes consist precisely of the lines through the origin (with the origin deleted).
The real projective space $\mathbb{R}P^n$ is the quotient space $(\mathbb{R}^{n+1} \setminus 0)/\sim$.

Remarks

1. We denote by $[a^0, \ldots, a^n]$ the class of $(a^0, \ldots, a^n) \in \mathbb{R}^{n+1}/\sim$.
2. We call $[a^0, \ldots, a^n]$ homogeneous coordinates on $\mathbb{R}P^n$.
3. We also let $\pi : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{R}P^n$ be the canonical projection.
Remark

1. Every line in \( \mathbb{R}^{n+1} \) through the origin meets the unit sphere \( S^{n+1} \) at a pair of antipodal points.

2. Conversely, there is a unique line through the origin and two antipodal points of \( S^{n+1} \)

Fig. 7.5. A line through the origin in \( \mathbb{R}^3 \) corresponds to a pair of antipodal points on \( S^2 \).
Real Projective Space

Facts

- On $S^{n+1}$ we define an equivalence relation by

$$x \sim y \iff x = \pm y.$$  

- The restriction of the canonical projection $\pi|_{S^n} : S^n \to \mathbb{R}P^n$ induces a continuous map $\overline{\pi} : S^n/\sim \to \mathbb{R}P^n$.

- The continuous map $f : \mathbb{R}^{n+1} \setminus 0 \to S^{n+1}$, $x \to \frac{x}{\|x\|}$ induces a continuous map $\overline{f} : \mathbb{R}P^n \to S^n/\sim$.

- The maps $\overline{\pi} : S^n/\sim \to \mathbb{R}P^n$ and $\overline{f} : \mathbb{R}P^n \to S^n/\sim$ are inverses of each other.

Proposition (Exercise 7.11)

The real projective space $\mathbb{R}P^n$ is homeomorphic to the quotient space $S^n/\sim$. 
Example (Real projective line $\mathbb{RP}^1$; see also Example 7.12)

- If we regard as the unit circle $S^1$ as a subset of $\mathbb{C}$, then the map $S^1 \to S^1$, $z \to z^2$ induces a continuous map $S^1/\sim \to S^1$.
- This is a continuous bijection between compact spaces, and hence this is a homeomorphism (by Corollary A.36).
- Here $S^1/\sim$ is compact, since this is the image of $S^1$ by the canonical projection map $S^1 \to S^1/\sim$, which is continuous.
- We thus have a sequence of homeomorphisms,

$$\mathbb{RP}^1 \cong S^1/\sim \cong S^1.$$
Proposition (Proposition 7.14)

The equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus 0$ is an open equivalence relation.

Corollary (Corollary 7.15)

The real projective space $\mathbb{R}P^n$ is second countable.

Corollary (Corollary 7.16)

The real projective space $\mathbb{R}P^n$ is Hausdorff.
The Standard Differentiable Structure of $\mathbb{R}P^n$

Facts

- For $i = 0, \ldots, n$, define
  \[ U_i = \{ [a^0, \ldots, a^n] \in \mathbb{R}P^n; \ a^i \neq 0 \} . \]

- As the property $a^i \neq 0$ remains unchanged when we replace $(a^0, \ldots, a^n)$ by $(ta^0, \ldots, ta^n)$ with $t \neq 0$, we see that $U_i$ is well defined.

- We have $\pi^{-1}(U_i) = \pi^{-1}(\tilde{U}_i)$, where
  \[ \tilde{U}_i = \{ (a^0, \ldots, a^n) \in \mathbb{R}^{n+1}; a^i \neq 0 \} . \]

- As $\tilde{U}_i$ is an open set in $\mathbb{R}^{n+1} \setminus 0$, this shows that $U_i$ is an open set in $\mathbb{R}P^n$. 
Facts

- Define $\tilde{\phi}_i : \tilde{U}_i \rightarrow \mathbb{R}^n$ by

$$
\tilde{\phi}_i(a^0, \ldots, a^n) = \left( \frac{a^0}{a^i}, \ldots, \frac{a^{i-1}}{a^i}, \frac{a^{i+1}}{a^i}, \ldots, \frac{a^n}{a^i} \right).
$$

- As $\tilde{\phi}_i(ta^0, \ldots, ta^n) = \tilde{\phi}_i(a^0, \ldots, a^n)$ for all $t \neq 0$, the map $\tilde{\phi}_i$ induces a map $\phi_i : U_i \rightarrow \mathbb{R}^n$ such that

$$
\phi \left( [a^0, \ldots, a^n] \right) = \tilde{\phi}_i(a^0, \ldots, a^n),
$$

$$
= \left( \frac{a^0}{a^i}, \ldots, \frac{a^{i-1}}{a^i}, \frac{a^{i+1}}{a^i}, \ldots, \frac{a^n}{a^i} \right).
$$

- As $\tilde{\phi}_i : \tilde{U}_i \rightarrow \mathbb{R}^n$ is a continuous map, the induced map $\phi_i : U_i \rightarrow \mathbb{R}^n$ is continuous as well.
The Standard Differentiable Structure of $\mathbb{R}P^n$

Facts

- The map $\phi_i : U_i \to \mathbb{R}^n$ is a bijection with inverse $\psi_i : \mathbb{R}^n \to U_j$, where

  $$\psi_i(x^1, \ldots, x^n) = [x^1, \ldots, x^i, 1, x^{i+1}, \ldots, x^n].$$

- The inverse map $\psi_i = \phi_i^{-1}$ is continuous, since $\psi_i = \pi \circ \tilde{\psi}_i$, where $\tilde{\psi}_i : \mathbb{R}^n \to \tilde{U}_i$ is the continuous map given by

  $$\tilde{\psi}_i(x^1, \ldots, x^n) = (x^0, \ldots, x^i, 1, x^{i+1}, \ldots, x^n).$$

- Thus, the map $\phi_i : U_i \to \mathbb{R}^n$ is a homeomorphism.
The Standard Differentiable Structure of $\mathbb{R}P^n$

Facts

- We have

\[ \phi_0(U_0 \cap U_1) = \left\{ \left( \frac{a^1}{a^0}, \ldots, \frac{a^n}{a^0} \right) ; a^j \in \mathbb{R}, a^0 \neq 0, \ a^1 \neq 0 \right\} \]

\[ = \left\{ (x^1, \ldots, x^n) \in \mathbb{R}^n ; x^1 \neq 0 \right\}. \]

- The transition map $\phi_1 \circ \phi_0^{-1} : \phi_0(U_0 \cap U_1) \rightarrow \mathbb{R}^n$ is given by

\[ \phi_0 \circ \phi_1^{-1}(x^1, \ldots, x^n) = \phi_0 \left( [1, x^1, \ldots, x^n] \right), \]

\[ = \left( \frac{1}{x^1}, \frac{x^2}{x^1}, \ldots, \frac{x^n}{x^1} \right). \]

- In particular, this is a $C^\infty$ map.

- It can be similarly shown that all the other transition maps $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \mathbb{R}^n$ are $C^\infty$ maps.
The collection \( \{(U_i, \phi_i)\}_{i=0}^{n} \) is a \( C^\infty \) atlas for \( \mathbb{R}P^n \), and so \( \mathbb{R}P^n \) is a smooth manifold.

Definition

The differentiable structure defined by the atlas \( \{(U_i, \phi_i)\}_{i=0}^{n} \) is called the standard differentiable structure of \( \mathbb{R}P^n \).
We also define complex projective spaces.

- On $\mathbb{C}^{n+1}$ consider the equivalence relation
  \[ x \sim y \iff \exists \lambda \in \mathbb{C} \setminus 0 \text{ such that } x = \lambda y. \]
  In other words $x \sim y$ if and only if $x$ and $y$ lie on the same complex line through the origin.

- The equivalence classes are the complex lines through the origin (minus the origin).

- The \textit{complex projective space} $\mathbb{CP}^n$ is the quotient space $(\mathbb{C}^{n+1} \setminus 0)/\sim$.

- The class of $a = (a^0, \ldots, a^n)$ is denoted $[a^0, \ldots, a^n]$. We call $[a^0, \ldots, a^n]$ \textit{homogeneous coordinates}.

- The space $\mathbb{CP}^n$ is Hausdorff and 2nd countable.
Differentiable Structure on $\mathbb{C}P^n$

**Facts**

- For $i = 1, \ldots, n$, define
  \[ U_i = \{ [a^0, \ldots, a^n]; (a^0, \ldots, a^n) \in \mathbb{C}^{n+1} \setminus 0, \ a^i \neq 0 \} . \]
  
  This is an open set in $\mathbb{C}P^n$.
- Define $\phi_i : U_i \to \mathbb{C}^n$ by
  \[ \phi_i ([a^0, \ldots, a^n]) = \left( \frac{a^0}{a^i}, \ldots, \frac{a^{i-1}}{a^i}, \frac{a^{i+1}}{a^i}, \ldots, \frac{a^n}{a^i} \right) . \]
  
  This is a homeomorphism from $U_i$ on $\mathbb{C}^n$. It has inverse
  \[ \psi_i(z^1, \ldots, z^n) = (z^1, \ldots, z^i, 1, z^{i+1}, \ldots, z^n) . \]
- The transition maps $\phi_i \circ \phi_j^{-1}$ are $C^\infty$ maps (they even are holomorphic maps).
- Thus, $\{(U_i, \phi_i)\}_{i=1}^n$ is a $C^\infty$ atlas for $\mathbb{C}P^n$, and so the complex projective space $\mathbb{C}P^n$ is a manifold.