Commutative Algebra
Chapter 3: Rings and Modules of Fractions

Sichuan University, Fall 2020
The Field of Fractions of an Integral Domain

Reminder
We say that a ring $A$ is an integral domain when it has no non-zero divisors, i.e.,

$$xy = 0 \iff x = 0 \text{ or } y = 0.$$ 

Fact
In the same way we construct the rational field $\mathbb{Q}$ from the ring of integers $\mathbb{Z}$, with any integral domain $A$ we can associate its field of fractions $\text{Frac}(A)$. 
The Field of Fractions of an Integral Domain

Facts

Let \( A \) is an integral domain. Set \( S = A \setminus \{0\} \). On \( A \times S \) define a relation \( \equiv \) by

\[
(a, s) \equiv (b, t) \iff at = bs.
\]

- This relation is reflexive and symmetric,

\[
(a, s) \equiv (a, s), \quad (a, s) \equiv (b, t) \iff (b, t) \equiv (a, s).
\]

- To check transitivity, suppose that \( (a, s) \equiv (b, t) \) and \( (b, t) \equiv (c, u) \), i.e., \( at = bs \) and \( bu = ct \). Then

\[
t(au - cs) = (at)u - (ct)s = (bs)u - (bu)s = 0.
\]

- As \( t \neq 0 \) and \( A \) is an integral domain, this implies that \( au = cs \), i.e., \( (a, s) \equiv (c, u) \).

- Therefore, the relation \( \equiv \) is an equivalence relation on \( A \times S \).
The Field of Fractions of an Integral Domain

**Definition**
1. The class of \((a, s)\) is denoted by \(a/s\).
2. The set of equivalence classes is denoted by \(\text{Frac}(A)\).

**Proposition**
1. \(\text{Frac}(A)\) is a field with respect to the addition and multiplication given by
   \[
   \frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.
   \]
2. The map \(A \ni a \mapsto a/1 \in \text{Frac}(A)\) is an injective ring homomorphism, and hence embeds \(A\) as a subring into \(\text{Frac}(A)\).

**Definition**
The field \(\text{Frac}(A)\) is called the *field of fractions* of \(A\).
Examples

1. If $A = \mathbb{Z}$, then $\text{Frac}(A) = \mathbb{Q}$.

2. If $A$ is a polynomial ring $k[x]$, $k$ field, then $\text{Frac}(A)$ is the field of rational functions over $k$.

3. If $A$ is the ring of holomorphic functions on an open $\Omega \subset \mathbb{C}$, then $\text{Frac}(A)$ is the field of meromorphic functions on $\Omega$. 
Remark

- The construction of the field Frac(A) uses the fact that A is an integral domain.
- It still can be adapted for arbitrary rings.

In what follows we let A be a ring.

Definition

A subset S of A is called *multiplicatively closed* when

\[ 1 \in S \quad \text{and} \quad x, y \in S \implies xy \in S. \]

Example

The ring A is an integral domain if and only if \( A \setminus \{0\} \) is multiplicatively closed.
Rings of Fractions

Facts

Let $S$ be a multiplicatively closed subset of $A$. On $A \times S$ define a relation $\equiv$ by

$$(a, s) \equiv (b, t) \iff \exists u \in S \text{ such that } (at - bs)u = 0.$$  

- This relation is reflexive and symmetric.
- To check transitivity, suppose that $(a, s) \equiv (b, t)$ and $(b, t) \equiv (c, u)$, i.e., there are $v, w \in S$ such that

$$(at - bs)v = (bu - ct)w = 0.$$  

Then $(au - cs)tvw$ is equal to


As $S$ is multiplicatively closed, $tvw \in S$, and so $(a, s) \equiv (c, u)$.
- Thus, we have an equivalence relation on $A \times S$.  

Rings of Fractions

Definition
1. The class of \((a, s)\) is denoted by \(a/s\).
2. The set of equivalence classes is denoted by \(S^{-1}A\).

Proposition
1. \(S^{-1}A\) is a ring with respect to the addition and multiplication given by
   \[
   (a/s) + (b/t) = (at + bs)/st, \quad (a/s) \cdot (b/t) = ab/st.
   \]
2. The map \(f : A \to S^{-1}A, a \to a/1\) is a ring homomorphism.

Remarks
1. The ring homomorphism \(f : A \to S^{-1}A\) is not injective in general.
2. If \(A\) is an integral domain and \(S = A \setminus \{0\}\), then \(S^{-1}A\) is the field of fractions \(\text{Frac}(A)\).
Rings of Fractions

**Definition**

The ring $S^{-1}A$ is called the *ring of fractions* of $A$ with respect to $S$.

**Proposition (Universal Property of $S^{-1}A$; Proposition 3.1)**

Let $g : A \rightarrow B$ be a ring homomorphism such that $g(s)$ is a unit in $B$ for all $s \in S$. Then there is a unique ring homomorphism $h : S^{-1}A \rightarrow B$ such that $g = h \circ f$. 
Fact

The ring $S^{-1}A$ and the homomorphism $f : A \to S^{-1}A$ satisfy the following properties:

(i) $f(s)$ is a unit in $S^{-1}A$ for all $s \in S$.

(ii) If $f(a) = 0$, then $as = 0$ for some $s \in S$.

(iii) Every element of $S^{-1}A$ is of the form $f(a)f(s)^{-1}$ with $a \in A$ and $s \in S$.

Corollary (Corollary 3.2)

Let $B$ be a ring and $g : A \to B$ a ring homomorphism satisfying the properties (i)–(iii) above. Then there is a unique ring isomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$. 
Examples of Rings of Fractions

Example

- The single set $S = \{0\}$ is multiplicatively closed.
- In this case $S^{-1}A$ is the zero ring, since $(a, 0) \equiv (0, 0)$ for all $a \in A$.
- In fact, we have

$$S^{-1}A \text{ is the zero ring } \iff 0 \in S.$$ 

Example

Let $\alpha$ be an ideal in $A$, and set

$$S = 1 + \alpha = \{1 + x; \ x \in \alpha\} = \{x \in A; x = 1 \mod \alpha\}.$$ 

Then $S$ is multiplicatively closed.
Examples of Rings of Fractions

Example

Let \( f \in A \) and set \( S = \{ f^n; \ n \geq 0 \} \).

- The subset \( S \) is multiplicatively closed.
- We write \( A_f \) for \( S^{-1}A \) in this case.
- If \( A = \mathbb{Z} \) and \( f = q \in \mathbb{Z} \), then \( A_f \) consists of rational numbers of the form \( mq^{-n} \) with \( m \in \mathbb{Z} \) and \( n \geq 0 \).
Examples of Rings of Fractions

Reminder

- An ideal $p$ of $A$ is called a prime ideal when
  \[ xy \in p \iff x \in p \text{ or } y \in p. \]
- Any maximal ideal is prime.
- A local ring is a ring that has a unique maximal ideal.

Example

Let $p$ be a prime ideal, and set $S = A \setminus p$. We have

- $p$ is prime $\iff$ $S$ is multiplicatively closed.

We denote by $A_p$ the ring $S^{-1}A$ in this case.
### Facts

Let $m$ be the subset of $A_p$ consisting of elements of the form $a/s$ with $a \in p$ and $s \in S$.

- $m$ is an ideal of $A_p$.
- If $b/t \not\in m$, then $b \not\in p$, i.e., $b \in S$, and so $b/t$ is a unit in $A_p$ (with inverse $t/b$).
- Thus, if $a$ is an ideal such that $a \not\subseteq m$, then $a$ contains a unit, and hence $a = A$.
- It follows that $m$ is a maximal ideal of $a$ and is the only such ideal. Thus, $A_p$ is a local ring.

### Definition

The ring $A_p$ is called the localization of $A$ at $p$. 

Examples of Rings of Fractions

**Example**

\( A = \mathbb{Z} \) and \( p = (p) \), where \( p \) is a prime number. Then \( \mathbb{Z}_p \) consists of all rational numbers of the form \( \frac{m}{n} \) where \( n \) is prime to \( p \).

**Example**

\( A = k[t_1, \ldots, t_n] \), where \( k \) is a field, and \( p \) is a prime ideal in \( A \).

- \( A_p \) consists of all rational functions \( \frac{f}{g} \), where \( g \not\in p \).
- Let \( V \) be the variety defined by \( p \), i.e.,

\[
 p = \bigcap_{f \in p} f^{-1}(0) \subset k^n.
\]

If \( k \) is infinite, then \( A_p \) can be identified with the ring of all rational functions on \( k^n \) that are defined on almost all points of \( V \). It is called the *local ring of \( k^n \) along \( V \).*

- This is the prototype of local rings that arise in algebraic geometry.
The construction of $S^{-1}A$ can be further extended to $A$-modules.

**Facts**

Let $S$ be a multiplicatively closed subset of $A$ and $M$ an $A$-module. On $M \times S$ we define a relation $\equiv$ by

$$(m, s) \equiv (m, s') \iff \exists t \in S \text{ such that } t(s'm - sm') = 0.$$  

As before, this is an equivalence relation.

**Definition**

1. The equivalence class of $(m, s)$ is denoted $m/s$.
2. The set of equivalence classes is denoted $S^{-1}M$. 
Proposition

$S^{-1}M$ is an $S^{-1}A$-module with respect to the addition and scalar multiplication given by

$\left(\frac{m}{s}\right) + \left(\frac{m'}{s'}\right) = \frac{s'm + sm'}{ss'}$, \hspace{1cm} \left(\frac{a}{s}\right) \cdot \left(\frac{m}{t}\right) = \frac{am}{st}$.

Definition

$S^{-1}M$ is called the module of fractions of $M$ with respect to $S$. 
Fact

If \( u : M \to N \) is an \( A \)-module homomorphism, then we get an \( S^{-1}A \)-module homomorphism,

\[
S^{-1}u : S^{-1}M \longrightarrow S^{-1}N, \quad m/s \longrightarrow u(m)/s.
\]

Thus, the operation \( S^{-1} \) is a functor from the category of \( A \)-modules to the category of \( S^{-1}A \)-modules.

Proposition (Proposition 3.3)

The functor \( S^{-1} \) is exact, i.e., if \( M' \xrightarrow{f} M \xrightarrow{g} M'' \) is exact at \( M \), then \( S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M'' \) is exact at \( S^{-1}M \).
Remark
Let $M'$ be a sub-module of $M$.
- Applying the previous result to $M' \hookrightarrow M \twoheadrightarrow 0$ produces an injective $S^{-1}A$-module homomorphism $S^{-1}M' \rightarrow S^{-1}M$.
- This allows us to identify $S^{-1}M'$ with a sub-module of $S^{-1}M$. 
Corollary (Corollary 3.4)

Let \( N \) and \( P \) be sub-modules of \( M \). Then:

1. \( S^{-1}(N + P) = S^{-1}(N) + S^{-1}(P) \).
2. \( S^{-1}(N \cap P) = S^{-1}(N) \cap S^{-1}(P) \).
3. The \( S^{-1} \text{-modules} S^{-1}(M/N) \) and \( S^{-1}M/S^{-1}N \) are isomorphic.
We a canonical $A$-module isomorphism,

$$S^{-1}A \otimes_A M \simeq S^{-1}M, \quad (a/s) \otimes m \longrightarrow am/s.$$ 

Remarks

1. As $(a/s) \times m \rightarrow am/s$ is $A$-bilinear, by the universal property of the tensor product there is a unique $A$-module homomorphism $f : S^{-1}A \otimes_A M \rightarrow S^{-1}M$ such that

$$f((a/s) \otimes m) = am/s.$$

2. The $A$-module map $g : SM \rightarrow S^{-1}A \otimes_A M$, $m/s \rightarrow (1/s) \otimes m$ is an inverse of $f$, since

$$f \circ g(m/s) = f((1/s) \otimes m) = 1m/s = m/s,$$

$$g \circ f ((a/s) \otimes m) = g(am/s) = (1/s) \otimes am = (a/s) \otimes m.$$ 

Thus, $f : S^{-1}A \otimes_A M \rightarrow S^{-1}M$ is an $A$-module isomorphism.
Corollary (Corollary 3.6)

$S^{-1}A$ is a flat $A$-module, i.e., the functor $S^{-1}A \otimes -$ preserves exactness of $A$-module sequences.

Proposition (Proposition 3.7)

If $M$ and $N$ are $A$-modules, then we have a canonical isomorphism,

$$S^{-1}M \otimes_{S^{-1}A} S^{-1}N \simeq S^{-1}(M \otimes_A N), \quad (m/s) \otimes (n/t) \mapsto (m \otimes n)/st.$$ 

In particular, for any prime ideal $p$ of $A$ we get an $A_p$-module isomorphism,

$$M_p \otimes_{A_p} N_p \simeq (M \otimes_A N)_p.$$
The proof is similar to that of Proposition 3.5.

1. Due to the $S^{-1}A$-bilinearity of $(m/s) \times (n/t) \to (m \otimes n)/st$ there is a unique $S^{-1}A$-module homomorphism $f : S^{-1}M \otimes_{S^{-1}A} S^{-1}N \to S^{-1}(M \otimes A N)$ such that
   $$f ((m/s) \otimes (n/t)) = (m \otimes n)/st.$$ 

2. We also observe that
   $$(m/s) \otimes (n/t) = [(1/s)(m/1)] \otimes [(1/t)(n/1)] = \frac{1}{st} [(m/1) \otimes (n/1)].$$

   In particular, we have
   $$(m/st) \otimes (n/1) = \frac{1}{st} [(m/1) \otimes (n/1)] = (m/s) \otimes (n/t).$$

3. Using this it can be checked that $(m \otimes n)/s \to (m/s) \otimes (n/1)$ is an inverse of $f$, and hence $f$ is an isomorphism.
Local Properties

Definition

We say that a property $P$ of a ring $A$ (or an $A$-module $M$) is a local property when

$A$ (or $M$) has $P$ $\iff$ $A_p$ (or $M_p$) has $P$ for each prime ideal $p$ of $A$.

The next propositions provide examples of local properties.

Proposition (Proposition 3.8)

Let $M$ be an $A$-module. Then TFAE:

1. $M = 0$.
2. $M_p = 0$ for each prime ideal $p$ of $A$.
3. $M_m = 0$ for each maximal ideal $m$ of $A$. 
Local Properties

Proposition (Proposition 3.9; 1st Part)

Let $\phi : M \to N$ be an $A$-module homomorphism. Then TFAE:

1. $\phi$ is injective.
2. $\phi_p : M_p \to N_p$ is injective for every prime ideal $p$ of $A$.
3. $\phi_m : M_m \to N_m$ is injective for every maximal ideal $m$ of $A$.

Proposition (Proposition 3.9; 2nd Part)

Let $\phi : M \to N$ be an $A$-module homomorphism. Then TFAE:

1. $\phi$ is surjective.
2. $\phi_p : M_p \to N_p$ is surjective for every prime ideal $p$ of $A$.
3. $\phi_m : M_m \to N_m$ is surjective for every maximal ideal $m$ of $A$. 
As the following result shows, flatness is a local property.

**Proposition (Proposition 3.10)**

Let $M$ be an $A$-module. TFAE:

1. $M$ is a flat $A$-module.
2. $M_p$ is a flat $A_p$-module for every prime ideal $p$ of $A$.
3. $M_m$ is a flat $A_m$-module for every maximal ideal $m$ of $A$. 
Let $f : A \rightarrow B$ be a ring homomorphism.

- If $\mathfrak{a}$ is an ideal in $A$, then its extension $\mathfrak{a}^e$ is the ideal in $B$ generated by $f(\mathfrak{a})$. Thus, it consists of all finite sums,

$$\sum f(a_i)b_i, \quad a_i \in \mathfrak{a}, \quad b_i \in B.$$ 

- If $\mathfrak{b}$ is an ideal in $B$, then its contraction $\mathfrak{b}^c$ is the ideal $f^{-1}(\mathfrak{b})$ in $A$.

- If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in $A$, their ideal quotient is the ideal

$$(\mathfrak{a} : \mathfrak{b}) = \{ x \in A; xb \subseteq \mathfrak{a} \}.$$ 

When $\mathfrak{b} = (b)$ we write $(a : b)$ for $(a : (b))$. 


Extended and Contracted Ideals in Rings of Fractions

**Facts**

Let \( f : A \to S^{-1}A \) be the natural homomorphism \( a \to a/1 \).

- If \( \alpha \) is an ideal in \( A \), then any \( y \in \alpha^e \) is of the form
  \[
  y = \sum f(a_i)(b_i/s_i) = \sum (a_i/1)(b_i/s_i) = \sum a_i b_i / s_i,
  \]
  where \( a_i \in \alpha \), \( b_i \in B \) and \( s_i \in S \).
- Set \( s = \prod s_i \) and \( t_i = \prod_{j \neq i} s_j \), so that \( 1/s_i = t_i/s \). Then
  \[
  y = \sum (a_i b_i t_i / s) = \left( \sum a_i b_i t_i \right) / s.
  \]
- Set \( \alpha' = \sum a_i b_i t_i \). Then \( \alpha' \in \alpha \), and so \( y = \alpha' / s \in S^{-1} \alpha \).
- We then deduce that
  \[
  \alpha^e = S^{-1} \alpha.
  \]
Proposition (Proposition 3.11)

1. If \( q \) is an ideal in \( S^{-1}A \), then \( q = S^{-1}(q^c) \). Thus, any ideal in \( S^{-1}A \) is an extended ideal.

2. If \( \alpha \) is an ideal in \( A \), then \( \alpha^{ec} = \bigcup_{s \in S} (\alpha : s) \). In particular, \( \alpha^e = (1) \) if and only if \( S \cap \alpha \neq \emptyset \).

3. An ideal \( \alpha \) in \( A \) is a contracted ideal if and only if no element of \( S \) is a zero-divisor in \( A/\alpha \).

4. We have a one-to-one correspondence \( p \leftrightarrow S^{-1}p \) between prime ideal in \( S^{-1}A \) and prime ideals in \( A \) that don’t meet \( S \).

5. The operation \( S^{-1} \) on ideals commutes with taking finite sums, products, intersections, and radicals.
Reminder

- The *nilradical* of $A$ is the ideal

$$\mathfrak{N} = \{x \in A; \ x^n = 0 \text{ for some } n \geq 1\}.$$

- Equivalently, $\mathfrak{N}$ is the intersection of all the prime ideals of $A$ (see Proposition 1.8).

Corollary (Corollary 3.12)

*The nilradical of $S^{-1}A$ is precisely $S^{-1}\mathfrak{N}$.***
Corollary (Corollary 3.13)

Let \( p \) be a prime ideal of \( A \). Then the prime ideals of the local ring \( A_p \) are in one-to-one correspondence with the prime ideals of \( A \) that are contained in \( p \).

Remarks

- By this corollary, passing from \( A \) to \( A_p \) cuts out all prime ideals except those contained in \( p \).
- By Proposition 1.1, passing from \( A \) to \( A/p \) cuts out all prime ideals except those containing \( p \).
- Thus, if \( q \) is a prime ideal contained in \( p \), then passing to \( (A_p)/q \simeq (A/q)_p \) restricts ourselves to those prime ideals between \( q \) and \( p \).
- For \( q = p \) we obtain the residual field of \( p \). It can be realized either as the fraction field of the integral domain \( A/p \), or as the residue field of the local ring \( A_p \).
Reminder

- If $N$ and $P$ are sub-modules of an $A$-module $M$, then
  \[(N : P) = \{ x \in A; xP \subseteq N \}.\]
  This is an ideal of $A$.

- The \textit{annihilator} of $M$, denoted $\text{Ann}(M)$, is the ideal $(0 : M)$. That is,
  \[\text{Ann}(M) = \{ x \in A; xM = 0 \}.\]

- By Exercise 2.2 we have
  \[
  \text{Ann}(N + P) = \text{Ann}(N) \cap \text{Ann}(P),
  
  (N : P) = \text{Ann}((N + P)/N).
  \]
Proposition (Proposition 3.14)

Let $M$ be a finitely generated $A$-module. Then

$$S^{-1}(\text{Ann}(M)) = \text{Ann}(S^{-1}M).$$

Remark

- If $M$ is single generated, i.e., $M = Ax$. Then we have an exact sequence of $A$-modules,

$$0 \longrightarrow \text{Ann}(M) \longrightarrow A \xrightarrow{a \mapsto ax} M \longrightarrow 0.$$  

- By exactness of the functor $S$ this gives an exact sequence of $S^{-1}A$-modules,

$$0 \longrightarrow S^{-1}(\text{Ann}(M)) \longrightarrow S^{-1}A \xrightarrow{a/s \mapsto ax/s} S^{-1}M \longrightarrow 0,$$

which shows that $\text{Ann}(S^{-1}M) = S^{-1}(\text{Ann}(M)).$
Corollary (Corollary 3.15)

If $N$ and $P$ are sub-modules of $M$ with $P$ finitely generated, then

$$S^{-1}(N : P) = (S^{-1}N : S^{-1}P).$$

Remarks

- The fact that $P$ is finitely generated implies that $(N + P)/N$ is finitely generated as well.

- As $(N : P) = \text{Ann}((N + P)/N)$ by applying the previous proposition we get

$$S^{-1}(N : P) = \text{Ann} \left[ S^{-1}((N + P)/N) \right].$$

- We have

$$S^{-1}((N + P)/P) = S^{-1}(N + P)/S^{-1}N = (S^{-1}N + S^{-1}P)/S^{-1}N.$$

- Thus,

$$S^{-1}(N : P) = \text{Ann} \left[ (S^{-1}N + S^{-1}P)/S^{-1}N \right] = (S^{-1}N : S^{-1}P).$$
Proposition (Proposition 3.16)

Let \( g : A \to B \) be a ring homomorphism, and \( \mathfrak{p} \) a prime ideal in \( A \).
Then TFAE:

(i) \( \mathfrak{p} \) is the contraction of a prime ideal in \( B \).

(ii) \( \mathfrak{p}^{ec} = \mathfrak{p} \).