Notation

Throughout this chapter $A$ is a ring (which is commutative and has an identity element).

Definition (Modules)

An $A$-module is an Abelian group $M$ on which $A$ acts linearly. That is, there is a map $A \times M \ni (a, x) \to ax \in M$ such that

$$a(x + y) = ax + ay,$$

$$(a + b)x = ax + bx,$$

$$(ab)x = a(bx),$$

$$1x = x, \quad a, b \in A, \ x, y \in M.$$
## Examples

1. Any ideal $\alpha$ of $A$ is an $A$-module. In particular, $A$ itself is an $A$-module.

2. If $A$ is a field $k$, then an $A$-module is exactly a vector space of over $k$.

3. If $A = \mathbb{Z}$, then a $\mathbb{Z}$-module is just an Abelian group.

4. If $A = k[x]$, $k$ field, then an $A$-module is a $k$-vector space together with a linear transformation (which corresponds to the action of $x$).

5. If $A$ is the group ring $kG$ of a group $G$ over a field $k$, then an $A$-module is exactly a $k$-representation of $G$, i.e., a $k$-vector space together with a group morphism $G \to \text{End}(V)$.
Definition (Module Homomorphisms)

Given modules $M$ and $M'$, an map $f : M \rightarrow M'$ is an $A$-module homomorphism (or is $A$-linear) if

$$f(x + y) = f(x) + f(y),$$
$$f(ax) = af(x), \quad a \in A, \ x, y \in M.$$ 

Remark

In other words, $f$ is a homomorphism of Abelian groups that commutes with the action of $A$.

Example

If $A$ is a field $k$, then a $k$-module homomorphism is nothing but a linear transformation between $k$-vector spaces.
## Modules and Module Homomorphisms

### Definition

The set of all $A$-module homomorphisms $f : M \to M'$ is denoted by $\text{Hom}_A(M, N)$ (or simply $\text{Hom}(M, N)$ when there is no ambiguity on the ground ring).

### Fact

$\text{Hom}_A(M, N)$ is an $A$-module. Given $f, g \in \text{Hom}_A(M, N)$ and $a \in A$ we define $f + g$ and $af$ by

\[
(f + g)(x) = f(x) + g(x),
\]

\[
f(ax) = af(x), \quad x \in M.
\]
Remarks

1. The composition of $A$-module homomorphisms is again an $A$-module homomorphism.

2. Given homomorphisms $u : M \to M'$ and $v : N \to N'$ we get maps,

$$\tilde{u} : \text{Hom}_A(M, N) \to \text{Hom}_A(M', N), \quad \tilde{u}(f) = f \circ u,$$

$$\tilde{v} : \text{Hom}_A(M, N) \to \text{Hom}_A(M, N'), \quad \tilde{v}(f) = v \circ f.$$

These maps are $A$-module homomorphisms.

3. For any module $M$ we have a natural isomorphism,

$$\text{Hom}_A(A, M) \cong M.$$

Any $f \in \text{Hom}_A(A, M)$ is uniquely determined by $f(1)$. 
### Definition (Submodules)

A *submodule* $M'$ of a module $M$ is a subgroup that is closed under the action of $A$.

### Fact

If $M'$ is a submodule of $M$, then the Abelian group $M/M'$ inherits an $A$-module structure given by

$$a(x + M') = ax + M', \quad x \in M.$$  

### Definition (Quotient Modules)

$M/M'$ is called the *quotient of $M$ by $M'$*.

### Facts

1. The canonical map $M \to M/M'$ is an $A$-module homomorphism.
2. There is a one-to-one correspondance between submodules of $M$ that contains $M'$ and submodules of $M/M'$.  

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**Submodules and Quotient Modules**
Submodules and Quotient Modules

**Definition**

Let $f : M \rightarrow N$ be an $A$-module homomorphism.

1. The *kernel* of $f$ is
   \[
   \ker(f) = \{ x \in M ; \; f(x) = 0 \}.
   \]
   This is a submodule of $M$.

2. The *image* of $f$ is
   \[
   \text{im}(f) = f(M).
   \]
   This is a submodule of $N$.

3. The *cokernel* of $f$ is
   \[
   \text{coker}(f) = N / \text{im}(f).
   \]
   This is a quotient module of $N$. 


Submodules and Quotient Modules

Facts

Let $f : M \rightarrow N$ be an $A$-module homomorphism and $M'$ a submodule of $M$ such that $M' \subseteq \ker(f)$.

- $f$ gives rise to a homomorphism $\bar{f} : M/M' \rightarrow N$ defined by
  \[ \bar{f}(\overline{x}) = f(x), \]
  where $\overline{x} \in M/M'$ is the image of $x \in M$.
- The kernel of $\bar{f}$ is $\ker(f)/M'$.

Definition (Induced Homomorphisms)

The homomorphism $\bar{f}$ is said to be induced by $f$.

Remark

For $M' = \ker(f)$ we get an isomorphism,

\[ M/\ker(f) \cong \text{im}(f). \]
Operations on Modules

**Definition**
Let $M$ be an $A$-module, and $(M_i)_{i \in I}$ be a family of sub-modules of $M$. The *sum* $\sum M_i$ consists of all finite sums $\sum x_i$, where $x_i \in M_i$ and all but finitely of the $x_i$ are zero.

**Remark**
$\sum M_i$ is the smallest sub-module that contains all the $M_i$.

**Facts**
1. The intersection $\cap M_i$ is again a submodule of $M$.
2. The submodules form a lattice with respect to inclusion.
Proposition (Proposition 2.1)

1. If $L \supseteq M \supseteq N$ are $A$-modules, then

$$(L/M)/(M/N) \cong L/M.$$ 

2. If $M_1$ and $M_2$ are submodules of $M$, then

$$(M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2).$$
**Definition**

If $\mathfrak{a}$ is an ideal of $A$ and $M$ is an $A$-module, then $\mathfrak{a}M$ consists of all finite sums $\sum a_i x_i$ with $a_i \in \mathfrak{a}$ and $x_i \in M$.

**Remarks**

1. $\mathfrak{a}M$ is a submodule of $M$.
2. In general we cannot define the product of two submodules.
Operations on Modules

**Definition**

1. If $N$ and $P$ are submodules of $M$, then $(N : P)$ is the set of all $a \in A$ such that $aP \subseteq N$.
2. $(0 : M)$ is called the *annihilator* of $M$ and is denoted by $\text{Ann}(M)$.

**Remarks**

1. $(N : P)$ is an ideal of $A$.
2. $\text{Ann}(M)$ consists of all $a \in A$ such that $aM = 0$. 

Fact
If $\mathfrak{a}$ is an ideal contained in $\text{Ann}(M)$, then we may regard $M$ as an $A/\mathfrak{a}$-module as follows: if $m \in M$ and $\bar{x} \in A/\mathfrak{a}$ is the class of $x \in A$, then

$$\bar{x}m = xm.$$ 

This definition makes sense since $\mathfrak{a}M = 0$.

Definition (Faithful Modules)
We say that an $A$-module $M$ is faithful when $\text{Ann}(M) = 0$.

Remark
If $\mathfrak{a} = \text{Ann}(M)$, then $M$ is always faithful as an $A/\mathfrak{a}$-module.
### Exercise (Exercise 2.2)

(i) \( \text{Ann}(M + N) = \text{Ann}(M) \cap \text{Ann}(N). \)

(ii) \( (N : P) = \text{Ann}((N + P)/N). \)
**Fact**

If \( x \in M \), the set of all multiples \( ax \) with \( a \in A \) is a submodule of \( M \) denoted by \( Ax \) or \((x)\).

**Definition**

- If \( M = \sum A_{x_i} \), then we say that the \( x_i \) form a *set of generators* of \( M \).
- We say that \( M \) is *finitely generated* if it admits a finite set of generators.

**Remark**

- That the \( x_i \) form a set of generators of \( M \) means that every \( x \in M \) is a finite linear combination \( \sum a_i x_i \) with \( a_i \in A \).
- This linear combination need not be unique.
Definition (Direct Sum)

If $M$ and $N$ are $A$-modules, their direct sum $M \oplus N$ consist of all pairs $(x, y)$ with $x \in M$ and $y \in N$.

This is an $A$-module with respect to the following addition and scalar multiplication,

\[
(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),
\]
\[
a(x, y) = (ax, ay).
\]
Direct Sum and Product

Definition (Direct Sum and Direct Product)

Let \((M_i)_{i \in I}\) be a family of \(A\)-modules.

1. The \textit{direct sum} \(\bigotimes M_i\) consists of all families \((x_i)_{i \in I}\) where all but finitely many of the \(x_i\) are zero.

2. The \textit{direct product} \(\prod M_i\) consists of all families \((x_i)_{i \in I}\).

Remarks

1. The direct sum \(\bigotimes M_i\) and the direct product \(\prod M_i\) are both \(A\)-modules.

2. They agree when the index set \(I\) is finite.
Suppose that the ring $A$ is a direct product $\prod_{i=1}^{n} A_i$.

1. Let $a_i$ be the set of all elements of $a$ of the form

$$(0, \ldots, 0, a_i, 0, \ldots, 0), \quad a_i \in A.$$ 

This is an ideal of $A$.

2. The ring $A$, considered as an $A$-module, agrees with the direct sum $a_1 \oplus \cdots \oplus a_n$. 
Direct Sum and Product

Facts

Conversely, suppose we have a module decomposition,

\[ A = a_1 \oplus \cdots \oplus a_n, \]

where \( a_1, \ldots, a_n \) are ideals.

1. We have

\[ A \cong \prod_{i=1}^{n} (A/b_i), \quad \text{where} \quad b_i = \bigoplus_{j \neq i} a_j. \]

2. Each ideal \( a_i \) is a ring isomorphic to \( A/b_i \).

3. The identity element \( e_i \) of \( a_i \) is an idempotent in \( A \) and \( a_i = (e_i) \).
Definition (Free Modules)

A free $A$-module is an $A$-module of the form $\oplus_{i \in I} M_i$, where $M_i \cong A$.

Example

- The direct sum $A^n = A \oplus \cdots \oplus A$ ($n$ summands) is a free module.
- By convention $A^0$ is the zero module, denoted by 0.

Fact

Any finitely generated free module is isomorphic to $A^n$ for some $n$. 

Proposition (Proposition 2.3)

An $A$-module $M$ is finitely generated if and only if it a quotient of $A^n$ for some $n \geq 1$.

Proposition (Proposition 2.4; see Gaillard)

Suppose that $M$ is a finitely generated $A$-module and $\mathfrak{a}$ is ideal of $A$. Let $\phi : M \rightarrow M$ be an $A$-module endomorphism such that $\phi(M) \subseteq \mathfrak{a}M$. The $\phi$ satisfies an equation of the form,

$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0, \quad a_i \in \mathfrak{a}.$$  

Corollary (Corollary 2.5)

Let $M$ be a finitely generated $A$-module and $\mathfrak{a}$ an ideal of $A$ such that $\mathfrak{a}M = M$. Then there is $x \equiv 1 \mod \mathfrak{a}$ such that $xM = 0$. 
Proposition (Nakayama’s Lemma; Proposition 2.6)

Let \( M \) be a finitely generated \( A \) module and \( \mathfrak{a} \) an ideal of \( A \) contained in the Jacobson radical \( \mathfrak{R} \) of \( A \). Then \( \mathfrak{a}M = M \) implies that \( M = 0 \).

Corollary (Corollary 2.7)

Let \( M \) be a finitely generated \( A \) module, \( N \) a submodule of \( A \), and \( \mathfrak{a} \) an ideal of \( A \) contained in \( \mathfrak{R} \). Then \( M = \mathfrak{a}M + N \Rightarrow M = N \).
Finitely Generated Modules

Fact

Let $A$ be a local ring, $\mathfrak{m}$ its maximal ideal, and $k = A/\mathfrak{m}$ its residue field. Let $M$ be a finitely generated $A$-module $M$. Then

- The quotient module $V = M/\mathfrak{m}M$ is annihilated by $\mathfrak{m}$, and hence this is an $A/\mathfrak{m}$-module, i.e., a vector space over $k$.
- This vector space has finite dimension.

Proposition (Proposition 2.8)

Let $x_1, \ldots, x_n$ be elements in $M$ whose images in $M/\mathfrak{m}M$ form a basis of this vector space. Then $x_1, \ldots, x_n$ generate $M$. 
Definition (Exact Sequences)

A sequence of $A$-modules and $A$-homomorphisms

$$
\cdots \rightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \rightarrow \cdots
$$

is said to be exact at $M_i$ when $\text{im}(f_i) = \text{ker}(f_{i+1})$. It is called an exact sequence when it is exact at each $M_i$.

Examples

1. A sequence $0 \rightarrow M' \xrightarrow{f} M$ is exact if and only if $f$ is injective.
2. A sequence $M \xrightarrow{g} M'' \rightarrow 0$ is exact if and only if $g$ is surjective.
3. A sequence $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is exact if and only if $f$ is injective, $g$ is surjective, and $g$ induces an isomorphism of $\text{coker}(f) = M/f(M')$ onto $M''$. Such a sequence is called a short exact sequence.
**Remark**

- Any long exact sequence,

\[ \cdots \rightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \rightarrow \cdots \]

\[ \xrightarrow{f_i} \]

can be split up into short exact sequences.

- If we set \( N_i = \text{im}(f_i) = \ker(f_{i+1}) \), then, for each \( i \), we have a short exact sequence,

\[ 0 \rightarrow N_i \rightarrow M_i \xrightarrow{f_i} N_{i+1} \rightarrow 0. \]
Proposition (Proposition 2.9; see Carlson)

1. A sequence of the form

\[ M' \overset{u}{\longrightarrow} M \overset{v}{\longrightarrow} M'' \overset{\quad}{\longrightarrow} 0 \]

is exact if and only if, for every \( A \)-module \( N \), the sequence

\[ 0 \longrightarrow \text{Hom}(M'', N) \overset{\tilde{v}}{\longrightarrow} \text{Hom}(M, N) \overset{\tilde{u}}{\longrightarrow} \text{Hom}(M', N) \]

is exact.

2. A sequence of the form

\[ 0 \longrightarrow N' \overset{u}{\longrightarrow} N \overset{v}{\longrightarrow} N'' \]

is exact if and only if, for every \( A \)-module \( M \), the sequence

\[ 0 \longrightarrow \text{Hom}(M, N') \overset{\tilde{u}}{\longrightarrow} \text{Hom}(M, N) \overset{\tilde{v}}{\longrightarrow} \text{Hom}(M, N'') \]

is exact.
Proposition (Snake Lemma, Proposition 2.10; see Carlson)

Suppose that

\[
\begin{array}{cccccccc}
0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\
& & f' & \downarrow & f & \downarrow & f'' & & \\
0 & \longrightarrow & N' & \xrightarrow{u'} & N & \xrightarrow{v'} & N'' & \longrightarrow & 0
\end{array}
\]

is a commutative diagram of $A$-modules and $A$-homomorphisms with exact rows. Then there exists an exact sequence,

\[
0 \longrightarrow \ker(f') \xrightarrow{\bar{u}} \ker(f) \xrightarrow{\bar{v}} \ker(f'') \xrightarrow{d} \\
\coker(f') \xrightarrow{\bar{u}'} \coker(f) \xrightarrow{\bar{v}'} \coker(f'') \longrightarrow 0,
\]

where $\bar{u}, \bar{v}$ are restrictions of $u, v$, and $\bar{u}', \bar{v}'$ are induced by $u', v'$. 
Remark

The map $d : \ker(f'') \to \coker(f')$ is called boundary homomorphism. It is constructed as follows:

- If $x'' \in \ker(f'')$, we have $x'' = v(x)$ for some $x \in M$ (since $v$ is surjective).
- We have $v'(f(x)) = f''(v(x)) = f(x'') = 0$, and so $f(x) \in \ker(v') = \text{ran}(u')$.
- As $u'$ is injective, there is a unique $y' \in N'$ such that $f(x) = u'(y')$.
- We define $dx''$ to be the class of $y'$ in $\coker(f') = N'/\text{im}(f')$.
- It can be shown that the class of $y'$ does not depend on the choice of $x$, and $dx''$ is well defined.
Definition

Let $\mathcal{C}$ be a class of $A$-modules. A function $\lambda : \mathcal{C} \to \mathbb{Z}$ is additive when, for every short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{C}$, we have

$$\lambda(M') - \lambda(M) + \lambda(M'') = 0.$$  

Example

Let $A$ be a field $k$ and $\mathcal{C}$ be the class of finite-dimension vector spaces over $k$. Then the function $V \to \dim V$ is additive.

Proposition (Proposition 2.11)

Let $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ be an exact sequence in $\mathcal{C}$. Then, for any additive function $\lambda$ on $\mathcal{C}$, we have

$$\sum_{0 \leq i \leq n} (-1)^i \lambda(M_i) = 0.$$
Definition (A-Bilinear Maps)

Given $A$-modules an \textit{A-bilinear map} is any map $f : M \times N \times P$ such that

(i) For every $x \in M$, the map $N \ni y \mapsto f(x, y) \in P$ is $A$-linear.

(ii) For every $y \in N$, the map $M \ni x \mapsto f(x, y) \in P$ is $A$-linear.

Fact

Given $A$-modules $M$ and $N$ their \textit{tensor product} $M \otimes_A N$ is an $A$-module such that $A$-bilinear maps $M \times N \times P$ are in one-to-one correspondance with $A$-linear maps $M \otimes_A N \rightarrow P$. 
Proposition (Proposition 2.12)

Let $M$, $N$ be $A$-modules.

1. There exist an $A$-module $M \otimes_A N$ and an $A$-bilinear map $\otimes : M \times N \to M \otimes_A N$ satisfying the following universal property:

   For any $A$-module $P$ and $A$-bilinear map $f : M \times N \to P$ there is a unique $A$-linear map $f' : M \otimes_A N \to P$ such that

   $$f(x, y) = f'(x \otimes y) \quad \text{for all } x \in M \text{ and } y \in N.$$

2. If $(M \otimes' N, \otimes')$ is another pair satisfying the above universal property, then there is a unique isomorphism $j : M \otimes_A N \to M \otimes' N$ such that

   $$j(x \otimes y) = x \otimes' y \quad \text{for all } x \in M \text{ and } y \in N.$$
Remark

The $A$-module $M \otimes_A N$ is constructed as follows:

- Let $C$ be the free $A$-module $\prod_{(x,y) \in M \times N} A$ generated by all pairs $(x, y) \in M \times N$. It consists of finite formal linear combinations $\sum a_j(x_j, y_j)$ with $a_j \in A$ and $(x_j, y_j) \in M \times N$.

- Let $D$ be the submodule generated by elements of the form,

  $$(x + x', y) - (x, y) - (x', y), \quad (x, y + y') - (x, y) - (x, y'),
  \quad (ax, y) - a(x, y), \quad (x, ay) - a(x, y).$$

- The $A$-module $M \otimes_A N$ is the quotient module $C/D$. If $(x, y) \in M \times N$, we denote by $x \otimes y$ the class of $(x, y)$ in $M \otimes_A N$. From the definition we have

  $$(x + x') \otimes y = x \otimes y - x' \otimes y, \quad x \otimes (y + y') = x \otimes y - x \otimes y',
  \quad (ax) \otimes y = a(x \otimes y), \quad x \otimes (ay) = a(x \otimes y).$$

  That is, $\otimes : M \times N \to M \otimes_A N$ is an $A$-bilinear map.
Remarks

1. We often denote $M \otimes_A N$ by $M \otimes N$ when the ring $A$ is understood from context.

2. In practice, we will not need the construction of the tensor product. What is essential is to keep in mind its universal property.

3. If $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ are generator sets of $M$ and $N$, respectively, then the elements $x_i \otimes y_j$ generate $M \otimes N$. In particular, if $M$ and $N$ are finitely generated, then $M \otimes N$ is finitely generated as well.
Remark

- Let $M'$ and $N'$ are submodules of $M$ and $N$, respectively, If $x \in M'$ and $y \in N'$, then it may happen that $x \otimes y$ is zero as an element of $M \otimes N$, but is not zero as an element of $M' \otimes N'$.

- Take $A = M = \mathbb{Z}$, $N = N' = \mathbb{Z}/2\mathbb{Z}$ and $M' = 2\mathbb{Z}$. Let $x$ be the non-zero element of $\mathbb{Z}/2\mathbb{Z}$. Then $2 \otimes x$ is not zero in $M' \otimes N'$ since it generates $M' \otimes N'$. However, it is zero in $M \otimes N$, since we have

$$2 \otimes x = 2(1 \otimes x) = 1 \otimes (2x) = 0.$$  

- Nevertheless, we have the following result:

Corollary (Corollary 2.13)

Let $x_i \in M$ and $y_i \in N$ be such that $\sum x_i \otimes y_i = 0$ in $M \otimes N$. Then there are finitely generated submodules $M_0$ of $M$ and $N_0$ of $N$ such that $\sum x_i \otimes y_i = 0$ in $M_0 \otimes N_0$. 
Remark

- We can also define multi-tensor products $M_1 \otimes \cdots \otimes M_r$ by using multilinear maps instead of
- A map $M_1 \times \cdots \times M_r \to P$ is multilinear when it is linear with respect to each argument.

Proposition (Proposition 2.12*; see Carlson)

1. There exist an $A$-module $M_1 \otimes \cdots \otimes M_r$ and an multilinear map $\otimes \cdots \otimes : M_1 \times \cdots \times M_r \to M_1 \otimes \cdots \otimes M_r$ satisfying the following universal property:

   For any $A$-module $P$ and multilinear map $f : M_1 \times \cdots \times M_r \to P$ there is a unique $A$-linear map $f' : M_1 \otimes \cdots \otimes M_r \to P$ such that

   $$f(x_1, \cdots, x_r) = f'(x_1 \otimes \cdots \otimes x_r) \quad \text{for all } x_i \in M_i.$$

2. The pair $(M_1 \otimes \cdots \otimes M_r, \otimes \cdots \otimes)$ is unique up to isomorphism.
Proposition (Proposition 2.14; see Atiyah-MacDonald and Carlson)

Let $M, N, P$ be $A$-modules. Then we have canonical isomorphisms:

(i) $M \otimes N \simeq N \otimes M$, where $x \otimes y \to y \otimes x$.

(ii) $(M \otimes N) \otimes P \simeq M \otimes (N \otimes P) \simeq M \otimes N \otimes P$, where

$$(x \otimes y) \otimes z \to x \otimes (y \otimes z) \to x \otimes y \otimes z.$$  

(iii) $(M \oplus N) \otimes P \simeq (M \oplus P) \otimes (N \oplus P)$, where

$$(x + y) \otimes z \to (x \otimes z) + (y \otimes z).$$

(iv) $A \otimes M \simeq A$, where $a \otimes x \to ax$. 
Definition (Bimodules)

Given rings $A$ and $B$, an $(A, B)$-bimodule is an Abelian group $N$ which is both an $A$-module and a $B$-module and the two structures are compatible in the sense that

$$a(xb) = (ax)b, \quad a \in A, \ x \in M, \ b \in B.$$ 

Exercise (Exercise 2.15; see Carlson)

Suppose that $M$ is an $A$-module, $P$ is an $B$-module, and $N$ is an $(A, B)$-bimodule. Then

- $M \otimes_A N$ is naturally a $B$-module.
- $N \otimes_B P$ is naturally an $A$-module.
- We have a natural isomorphism of $(A, B)$-bimodules,

$$(M \otimes_A N) \otimes_B P \simeq M \otimes_A (N \otimes_B P).$$
Tensor Product of Modules

Facts

- If $f : M \to M'$ and $g : N \to N'$ are $A$-linear maps, then $M \times N \ni (x, y) \mapsto f(x) \otimes g(y) \in M' \otimes N'$ is an $A$-bilinear map.

- Therefore, there is a unique $A$-linear map $f \otimes g : M \otimes N \to M' \otimes N'$ such that

  $$(f \otimes g)(x \otimes y) = f(x) \otimes g(y) \quad \text{for all } x \in M \text{ and } y \in N.$$ 

- If $f' : M' \to M''$ and $g' : N' \to N''$ are $A$-linear maps, then we have

  $$(f \circ f') \otimes (g \otimes g') = (f' \otimes g') \circ (f \otimes g)$$
Fact
Let \( f : A \to B \) be a ring homomorphism.

- Any \( B \)-module \( N \) can be turned into an \( A \)-module as follows: if \( a \in A \) and \( x \in N \), then \( ax \) is defined to be \( f(a)x \).
- This \( A \)-module is said to be obtained from \( N \) by restriction of scalars.
- In particular, \( B \) is a module over \( A \) this way,

Proposition (Proposition 2.16)
If \( N \) is finitely generated as a \( B \)-module and \( B \) is finitely generated as an \( A \)-module, then \( N \) is finitely generated as an \( A \)-module.
Fact
Let $M$ be an $A$-module.

- As $B$ is an $A$-module, we can form the tensor product $M_B = B \otimes_A M$.
- In fact, $M_B$ is a $B$-module such that
  \[
  b(b' \otimes x) = (bb') \otimes x, \quad b, b' \in B, \ x \in M.
  \]
- We say that $M_B$ is obtained from $M$ by extensions of the scalars.

Proposition (Proposition 2.17)
If $M$ is finitely generated as an $A$-module, then $M_B$ is finitely generated as a $B$-module.
Fact

Let $S$ be the set of all $A$-bilinear maps $f : M \times N \to P$.

- $S$ is an $A$-module.
- If $f : M \times N \to P$ is bilinear, then, for every $x \in M$, we have an $A$-linear map $y \mapsto f(x, y)$. It depends linearly on $x$ and $f$, and so we get an $A$-linear map $S \to \text{Hom}_A(M, \text{Hom}_A(N, P))$.
- Conversely, any $\phi \in \text{Hom}_A(M, \text{Hom}_A(N, P))$ gives rise an $A$-bilinear map $(x, y) \mapsto (\phi(x))(y)$.
- Therefore, we have a canonical isomorphism,

$$S \cong \text{Hom}_A(M, \text{Hom}_A(N, P)).$$
Fact

Thanks to the defining property of the tensor product we also have a canonical isomorphism,

\[ S \cong \text{Hom}_A(M \otimes N, P). \]

Consequence

We have a canonical isomorphism,

\[ \text{Hom}_A(M \otimes N, P) \cong \text{Hom}_A(M, \text{Hom}_A(N, P)). \]

Remark

In the language of functors on the category of \( A \)-modules, the above result means that the functor \( - \otimes_A N : M \to M \otimes N \) is the left adjoint of the functor \( \text{Hom}_A(N, -) : P \to \text{Hom}_A(N, P) \) (and hence \( \text{Hom}_A(N, -) \) is the right adjoint of \( - \otimes_A N \)).
Proposition (Proposition 2.18)

Suppose we are given an exact sequence of $A$-modules and homomorphisms of the form,

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0.$$

Then, for every $A$-module, we have an exact sequence,

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \rightarrow 0,$$

where we have denote by $1$ the identity map of $N$.

Remarks

1. The above result means that, for every $A$-module $N$, the functor $- \otimes_A N$ is left exact.

2. More generally, it can be shown that any functor that is a left adjoint is left exact.

3. Likewise, any functor that is a right adjoint is right exact.
Remark

The functor $- \otimes_A N$ need not be *exact*, i.e., if $M' \to M \to M''$ is an exact sequence, then the sequence $M' \otimes N \to M \otimes N \to M'' \otimes N$ may fail to be exact.

Example

Take $A = \mathbb{Z}$ and consider the exact sequence,

$$0 \longrightarrow \mathbb{Z} \overset{f}{\longrightarrow} \mathbb{Z}, \quad f(x) = 2x.$$

- If we take tensor products with $N = \mathbb{Z}/2\mathbb{Z}$, then the sequence $0 \to \mathbb{Z} \otimes N \overset{f \otimes 1}{\longrightarrow} \mathbb{Z}$ is not exact.
- Indeed, given any $x \in \mathbb{Z}$ and $y \in \mathbb{Z}/2\mathbb{Z}$, we have

$$(f \otimes 1)(x \otimes y) = f(x) \otimes y = 2x \otimes y = x \otimes (2y) = 0.$$  

- This implies that $(f \otimes 1) = 0$, i.e., $\text{ran}(f \otimes 1) = 0 \neq \mathbb{Z} \otimes N$. Therefore, the sequence is not exact.
Definition (Flat Module)

We say that an $A$-module is flat when the functor $- \otimes_A N$ is exact, i.e., for every exact sequence $M' \to M \to M''$, the sequence $M' \otimes N \to M \otimes N \to M'' \otimes N$ is again exact.

Proposition (Proposition 2.19; see also Gaillard)

Let $N$ be an $A$-module. Then TFAE:

(i) $N$ is flat.

(ii) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then so is the tensored sequence $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$.

(iii) If $f : M' \to M$ is injective, then so is $f \otimes 1 : M' \otimes N \to M \otimes N$.

(iv) If $f : M' \to M$ is injective and $M$ and $M'$ are finitely generated, then $f \otimes 1 : M' \otimes N \to M \otimes N$ is again injective.
Exercise (Exercise 2.20; see Carlson)

If $A \to B$ is a ring homomorphism, and $M$ is a flat $A$-module, then $M_B = B \otimes_A M$ is a flat $B$-module.

*Hint:* Use the isomorphisms from Proposition 2.14 and Exercise 2.15.
**Definition**

An *A-algebra* is an $A$-module $B$ together with a multiplication $(b, b') \rightarrow bb'$ which is $A$-bilinear and with respect to which $B$ is a ring.

**Remark**

The $A$-bilinearity of the multiplication means that

$$a(bb') = (ab)b' = b(ab') \quad a \in A, \ b, b' \in B.$$  

This accounts for the compatibility of the module and ring structures of $B$. 
Example

Let $B$ be a ring and let $f : A \to B$ be a ring homomorphism.

- By restriction $B$ is an $A$-module with $ab = f(a)b$.
- If $B$ is commutative or the image of $f$ is contained in the center of $B$, then $B$ is an $A$-algebra, since

\[ f(a)(bb') = (f(a)b)b' = b(f(a)b'), \quad a \in A, \quad b, b' \in B. \]
Fact

Let $B$ an $A$-algebra with an identity element $1_B$, and define $f : A \rightarrow B$ by

$$f(a) = a1_B, \quad a \in A.$$  

Then $f$ is a ring homomorphism whose image is contained in the center of $A$, since

$$(a1_B)(a'1_B) = a \left[ 1_B(a'1_B) \right] = a \left[ a'(1_B1_B) \right] = (aa')1_B,$$

$$(a1_B)b = a(1_Bb) = a(b1_B) = b(a1_B), \quad a, a' \in A, \ b \in B.$$
An $A$-algebra with identity is exactly a ring $B$ with identity together with a ring homomorphism $f : A \rightarrow B$ whose image is contained in the center of $B$.

A commutative $A$-algebra with identity is exactly a commutative ring $B$ with identity together with a ring homomorphism $f : A \rightarrow B$.

Remark

- In Atiyah-MacDonald’s book rings are assumed to be commutative and have an identity.
- Therefore, $A$-algebras are assumed to be commutative and have an identity element.
- Atiyah-MacDonald then defines an $A$-algebra as a ring $B$ together a ring homomorphism $f : A \rightarrow B$. 
Remark

Suppose that $A$ is a field $k$ and $B$ is a $k$-algebra with identity element.

- As $k$ is a field, the ring homomorphism $k \ni \lambda \to \lambda 1_B$ must be injective (see Proposition 1.2).
- Thus, $k$ can be identified as a subring of $B$.

Therefore, a $k$ algebra (with identity) is a ring that contains $k$ as a subring.

Examples

1. $A = k$ and $B = k[x_1, \ldots, x_n]$ (polynomials with $n$ variables).
2. $A = k$ and $B = kG$ (group ring of a group $G$). This is not a commutative algebra unless $G$ is Abelian.
Remark

If $B$ is a ring with identity, then $\mathbb{Z} \ni m \mapsto m1 \in B$ is a ring homomorphism. Therefore, any such ring is automatically a $\mathbb{Z}$-algebra.
### Definition (Algebra Homomorphisms)
Given $A$-algebras $B$ and $C$, a map $h : B \to C$ is called an $A$-algebra homomorphism when it is both a ring homomorphism and an $A$-module homomorphism.

### Remark
Suppose that $B$ and $C$ have identities with ring homomorphisms $f : A \to B$ and $g : A \to B$. For every $A$-algebra homomorphism $h : B \to C$, we have $g = h \circ f$. 
Definition (Finite Algebras)

A *finite* $A$-algebra is an $A$-algebra which it is finitely generated as an $A$-module.
Definition (Finitely Generated Algebras)

Let $B$ be a commutative $A$-algebra with identity. We say that $B$ is a \textit{finitely generated $A$-algebra} when there is a finite set of elements $x_1, \ldots, x_n$ such that every element of $B$ is a linear combination of monomials $x_1^{m_1} \cdots x_n^{m_n}$.

Remark

The above condition means that we have surjective $A$-algebra homomorphism $A[X_1, \ldots, X_n] \to B$ that maps $X_i$ to $x_i$. Every $b \in B$ is a polynomial in the generators $x_1, \ldots, x_n$.

Definition (Finitely Generated Rings)

A ring $A$ is said to be \textit{finitely generated} when it is finitely generated as a $\mathbb{Z}$-algebra.
**Proposition**

Let $B$ and $C$ be $A$-algebras. Then their tensor product $B \otimes_A C$ is an $A$-algebra with product such that

$$(b \otimes c)(b' \otimes c') = (bb') \otimes (cc'), \quad b, b' \in B, \quad c, c' \in C.$$ 

**Remark**

In general, we have

$$\left( \sum_i b_i \otimes c_i \right) \left( \sum_j b'_j \otimes c'_j \right) = \sum_{i,j} (b_i b'_j) \otimes (c_i c'_j).$$
Tensor Product of Algebras

Remark

The product of $B \otimes_A C$ is well defined:

- Consider the multilinear map,
  
  \[ B \times C \times B \times C \ni (b, c, b', c') \to (bb') \otimes (cc') \in B \otimes C. \]

- It gives rise to an $A$-module homomorphism,
  
  \[ B \otimes C \otimes B \otimes C \longrightarrow B \otimes C. \]

- As $B \otimes C \otimes B \otimes C \simeq (B \otimes C) \otimes (B \otimes C)$, we get an $A$-module homomorphism,
  
  \[ (B \otimes C) \otimes (B \otimes C) \longrightarrow B \otimes C. \]

- This then gives rise to an $A$-bilinear map,
  
  \[ (B \otimes C) \times (B \otimes C) \longrightarrow B \otimes C, \]

which is our product.
Remark

Suppose that $B$ and $C$ are commutative algebras with identity elements $1_B$ and $1_C$.

- The algebra $B \otimes C$ is a commutative and has $1_B \otimes 1_C$ as identity element.
- We have natural algebra homomorphisms,

$$
\begin{align*}
\text{id}_B \otimes 1_C & : B \rightarrow B \otimes C, \quad b \rightarrow b \otimes 1_C, \\
1_B \otimes \text{id}_C & : C \rightarrow B \otimes C, \quad c \rightarrow 1_B \otimes c.
\end{align*}
$$
Remark (Continued)

We actually have a commutative diagram,

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\uparrow & & \downarrow \text{id}_B \otimes 1_C \\
C & \rightarrow & B \otimes C \\
\downarrow & & \downarrow 1_B \otimes \text{id}_C \\
B & \rightarrow & C \\
\end{array}
\]