Noncommutative Geometry
Lecture 5: Diffeomorphism Invariant Geometry

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Diffeomorphism Invariant Geometry

Setup

- M^n is an oriented manifold.
- $\Gamma = \text{Diff}^+(M)$ is the group of all orientation-preserving diffeomorphisms of M.

Goal

Reformulate the index formula of Atiyah-Singer in diffeomorphism invariant geometry. To this end we shall

1. Construct a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ encoding the differential geometry of the action of Γ on M.
2. Derive an index formula for this spectral triple.

The Crossed-Product Algebra

The candidate for the algebra $\mathcal{A}$ of the spectral triple is the *crossed-product algebra*,

$$C_c^\infty(M) \rtimes \Gamma := \left\{ \text{finite sums} \sum_{\varphi \in \Gamma} f_\varphi U_\varphi;\ f_\varphi \in C_c^\infty(M) \right\},$$

where f_φ and U_φ are represented as operators such that

$$U_\varphi^* = U_\varphi^{-1} = U_{\varphi^{-1}}, \qquad U_\varphi f = (f \circ \varphi^{-1}) U_\varphi.$$

Remark

- M does carry any diffeomorphism invariant differentiable structure, except the manifold structure itself. In particular, it does not carry a diffeomorphismin invariant metric.
- We cannot represent the elements $\Gamma = \mathrm{Diff}^+(M)$ as unitary operators on the space of sections of a vector bundle of differential forms over M.
- We get a Γ-invariant metric by trading M for its metric bundle.

The Metric Bundle

Definition

The *metric bundle* $P \xrightarrow{\pi} M$ is the bundle of positive-definite 2-tensors $p_{ij} dx^i \otimes dx^j$.

Remark

If $\{x^j\}$ are coordinates of $x \in M$, then a point $p \in P_x$ has coordinates $\{x^j\}$ and (p_{ij}) with $p = p_{ij} x^i \otimes dx^j$. Using such coordinates, the action of Γ on P is given by

$$\varphi^*(x, p) = \left(\varphi(x), \left(\varphi'(x)^{-1}\right)^t (p_{ij}) \left(\varphi'(x)^{-1}\right)^t \right)$$

Remark

If $F^+(M)$ is the *positive frame bundle* of M, then

$$P = F^+(M)/\,\mathrm{SO}(n).$$

Γ-Invariant Bundles

Definition

The *vertical bundle* of the fibration $\pi : P \to M$ is

$$V := \ker d\pi \subset TP.$$

Definition

$\pi^* TM$ is the lift of TM to a vector bundle over P, so that

$$(\pi^* TM)_{(x,p)} = T_x M \qquad \forall (x,p) \in P.$$

Proposition

Both V and $\pi^* TM$ are Γ-invariant vector bundles.

Remark

We always have $TP \simeq (\pi^* TM) \oplus V$, but there is no Γ-invariant identification.

Γ-Invariant Metrics

Proposition (Connes-Moscovici)

1. *$\pi^* TM$ carries the Γ-invariant metric,*

$$g_H(x,p) := p_{ij} dx^i \otimes dx^j.$$

2. *V carries the Γ-invariant metric,*

$$g_V(x,p) = g_V(p) := \mathrm{Tr}\left[(p^{-1}dp) \otimes (p^{-1}dp) \right].$$

3. *$g := g_H \oplus g_V$ is a Γ-invariant metric on $(\pi^* TM) \oplus V$.*

4. *P carries the Γ-invariant volume form,*

$$\mathrm{vol}_g(x,p) = \sqrt{\det p}\, dx^1 \wedge \cdots \wedge dx^n \wedge \mathrm{vol}_{g_V}(p).$$

Remark

$g_V(p)$ is a $\mathrm{GL}_n(\mathbb{R})$-invariant metric on $P_n := \{p \in M_n(\mathbb{R});\ p > 0\}$.

Γ-Invariant Metrics

Proposition

The metrics g_H and g_V gives rise to

1. *A Γ-invariant Hermitian metric h on*
$$\Lambda(P) := (\pi^* \Lambda^*_{\mathbb{C}} T^* M) \otimes \Lambda^*_{\mathbb{C}} V.$$
2. *A Γ-invariant Hodge operator $\star : \Lambda(P) \to \Lambda(P)$ such that*
$$\star\alpha \wedge \beta = h(\alpha, \beta)\, \mathrm{vol}_g(x, p) \qquad \forall \alpha, \beta \in \Lambda(P).$$
3. *A Γ-invariant orthogonal $\mathbb{Z}_2$-grading,*
$$\Lambda(P) = \Lambda^+(P) \oplus \Lambda^-(P), \qquad \Lambda^\pm(P) := \{\alpha \in \Lambda(P);\ \star\alpha = \pm\alpha\}.$$

The Hilbert Space $L^2(P, \Lambda(P))$

Definition

The Hilbert space $L^2(P, \Lambda(P))$ is the completion of $C_c^\infty(P, \Lambda(P))$ with respect to the inner product,

$$\langle u, v \rangle := \int_P h(u(x,p), v(x,p)) \, \mathrm{vol}_g(x,p), \qquad u, v \in C_c^\infty(P, \Lambda(P)).$$

Remark

We have the orthogonal $\mathbb{Z}_2$-grading,

$$L^2(P, \Lambda(P)) = L^2\left(P, \Lambda^+(P)\right) \oplus L^2\left(P, \Lambda^-(P)\right).$$

This $\mathbb{Z}_2$-grading is Γ-invariant.

Remark

- We have a natural representation $\varphi \to U_\varphi$ of Γ in $L^2(P, \Lambda(P))$ defined by

$$U_\varphi u := \varphi^* u \qquad \forall u \in L^2(P, \Lambda(P)).$$

- As the inner product of $L^2(P, \Lambda(P))$ is Γ-invariant, this is a *unitary* representation. That is,

$$U_{\varphi^{-1}} = U_\varphi^{-1} = U_\varphi^* \qquad \forall \varphi \in \Gamma.$$

- Let $\varphi \in \Gamma$ and $f \in C_c^\infty(P)$. Then, for all $u \in L^2(P, \Lambda(P))$,

$$U_\varphi(fu) = \varphi^*(fu) = \varphi^* f \varphi^* u = f \circ \varphi^{-1} U_\varphi u.$$

  That is,

$$U_\varphi f = f \circ \varphi^{-1} U_\varphi.$$

Geometric Realization of $C_c^\infty(P) \rtimes \Gamma$

Definition

$$C_c^\infty(P) \rtimes \Gamma := \left\{ \text{finite sums} \sum_{\varphi \in \Gamma} f_\varphi U_\varphi ; \; f_\varphi \in C_c^\infty(M) \right\},$$

where U_φ is defined as in the previous slide.

Remarks

1. The crossed-product algebra $C_c^\infty(P) \rtimes \Gamma$ is thus realized as an algebra of bounded operators on $L^2(P, \Lambda(P))$.
2. The action of $C_c^\infty(P) \rtimes \Gamma$ on $L^2(P, \Lambda(P))$ preserves the $\mathbb{Z}_2$-grading,

$$L^2(P, \Lambda(P)) = L^2\left(P, \Lambda^+(P)\right) \oplus L^2\left(P, \Lambda^-(P)\right).$$

The Operator D

Aim

Construct an operator $D : C_c^\infty(P, \Lambda(P)) \to C_c^\infty(P, \Lambda(P))$ such that

1. D maps sections of $\Lambda^\pm(P)$ to sections of $\Lambda^\mp(P)$.
2. $[D, fU_\varphi]$ is bounded for all $f \in C_c^\infty(P)$ and $\varphi \in \Gamma$.
3. fD has compact resolvent for all $f \in C_c^\infty(P)$.

Remark

If we seek for an operator D as a ψDO, then the 2nd condition is tatamount to

1. D has order 1.
2. The 1st order part of D is Γ-invariant.

Overview of the Construction

- The operator D will be defined by an equation,

$$D|D| = Q.$$

  That is, $D = Q|Q|^{-\frac{1}{2}}$.
- The operator Q will be a differential operator of the form,

$$Q = Q_H + Q_V,$$

  where
  - Q_H is 1st order horizontal signature operator.
  - Q_V is a 2nd order vertical signature operator.
- As we shall see, the operator D is not a ψDO, but it lies in a pseudo-differential calculus of a different type.

The Vertical Operator Q_V

Definition

The *longitidunal differential* $d_V : C^\infty(P, \Lambda^\bullet V^*) \to C^\infty(P, \Lambda^\bullet V^*)$ is defined as follows:

- If $f \in C^\infty(P)$, then $d_V f := (df)_{|V}$.
- If $\alpha \in C^\infty(P, V^*)$, then, for all $X, Y \in C^\infty(P, V)$,

$$d_V\alpha(X, Y) := X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]).$$

- If $\alpha \in C^\infty(P, \Lambda^k V^*)$ and $\beta \in\in C^\infty(P, \Lambda^\bullet V^*)$, then

$$d_V(\alpha \wedge \beta) = (d_V\alpha) \wedge \beta + (-1)^k \alpha \wedge d_V\beta.$$

The Vertical Operator Q_V

Definition

The operator $Q_V : C^\infty(M, C^\infty(P, \Lambda^\bullet V^*) \to C^\infty(P, \Lambda^\bullet V^*)$ is

$$Q_V := d_V d_V^* - d_V^* d_V.$$

Remark

The operator Q_V is a Γ-invariant 2nd order differential operator. It can be thought of as a *2nd order vertical signature operator*.

The Horizontal Operator Q_H

Fact

The fibration $\pi : P \to M$ gives rise to a vector-bundle morphism,

$$\pi^* : \pi^*\Lambda^\bullet T^*M \longrightarrow \Lambda^\bullet T^*P.$$

Extra Datum

H is a horizontal connection, i.e., a subbundle $H \subset TP$ such that

$$TP = V \oplus H.$$

Remark

As $d\pi_{|H} : H \to \pi^*TM$ is a vector-bundle isomorphism, we get vector-bundle morphisms,

$$j_H : \pi^*TM \xrightarrow{(d\pi_{|H})^{-1}} H \hookrightarrow TP,$$

$$j_H^t : \Lambda^\bullet T^*P \longrightarrow \pi^*\Lambda^\bullet T^*M.$$

The Horizontal Operator Q_H

Definition

The *H-differential* $d_H : C^\infty(P, \pi^*\Lambda^\bullet T^*M) \to C^\infty(P, \pi^*\Lambda^\bullet T^*M)$ is defined by

$$d_H\alpha := j_H^t(d(\pi^*\alpha)) \qquad \forall \alpha \in C^\infty(P, \pi^*\Lambda^\bullet T^*M).$$

Definition

The operator $Q_H : C^\infty(P, \pi^*\Lambda^\bullet T^*M) \to C^\infty(P, \pi^*\Lambda^\bullet T^*M)$ is

$$Q_H := d_H + d_H^*.$$

Remarks

1. Q_H is a 1st order differential operator.
2. If H' is another horizontal connection, then $Q_{H'} = Q_H + R_V$, where R_V is 1st order vertical differential operator.

The Operator Q

Definition

The operator $Q : C^\infty(P, \Lambda(P)) \to C^\infty(\Lambda(P))$ is

$$Q := Q_H \otimes 1 + 1 \otimes Q_V.$$

Remarks

1. Q is a 2nd order differential operator mapping sections of $\Lambda^\pm(P)$ to $\Lambda^\mp(P)$.
2. Q is not elliptic, so it is not invertible in the classical ψDO-calculus. However, it is hypoelliptic, and as such it is invertible in the ψDO'-calculus.
3. If $\varphi \in \Gamma$, then $U_\varphi Q U_\varphi^* = Q + R_V$, where R_V is a first order *vertical* differential operator, and hence has lower order 1 in the ψDO'-calculus. Therefore, the 2nd order part of Q in the ψDO'-sense is Γ-invariant.

The Operator D

Definition

The operator $D : C_c^\infty(P, \Lambda(P)) \to C_c^\infty(P, \Lambda(P))$ is defined by the equation,

$$D|D| = Q.$$

Remarks

1. The operator D is a selfadjoint ψDO' of order 1.
2. D maps sections of $\Lambda^\pm(P)$ to $\Lambda^\mp(P)$.
3. If $\varphi \in \Gamma$, then $U_\varphi D U_\varphi^* - D$ has order 0 in the ψDO'-calculus. Thus, the 1st order part of D (in the ψDO'-sense) is Γ-invariant.

The Spectral Triple in Diff-Invariant Geometry

Theorem (Connes-Moscovici '95)

1. *The following is a spectral triple,*

$$\left(C_c^\infty(P) \rtimes \Gamma, L^2(P, \Lambda(P)), D\right),$$

with $L^2(P,\Lambda(P)) = L^2(P,\Lambda^+(P)) \oplus L^2(P,\Lambda^-(P))$.

2. *This spectral is* q*-summable with* $q = 2n + \frac{1}{2}n(n+1)$*, regular and has a discrete and simple dimension spectrum.*
3. *The CM cocycle makes sense and computes the index map,*

$$\operatorname{ind}_D[\mathcal{E}] = \langle \varphi_{CM}, \mathcal{E}\rangle \qquad \forall \mathcal{E} \in K_0(C_c^\infty(P) \rtimes \Gamma).$$

Remark

The direct calculation of the CM cocycle involves way too many terms to deal with, e.g., in the simplest case $M = S^1$ it amounts to 100 pages of computation.

Hopf Cyclic Cohomology

Proposition (Connes-Moscovici '98)

1. *There are a universal Hopf algebra $\mathcal{H}_n$, depending only on the dimension n, and a characteristic map,*

$$\theta : HC^*(\mathcal{H}_n, \mathrm{SO}(n)) \longrightarrow HC^*(C_c^\infty(P) \rtimes \Gamma),$$

where $HC^*(\mathcal{H}_n, \mathrm{SO}(n))$ is the relative Hopf cyclic cohomology of $\mathcal{H}_n$.

2. *The relative Hopf cyclic cohomology $HC^*(\mathcal{H}_n, \mathrm{SO}(n))$ is isomorphic to the Gel'fand-Fuks cohomology,*

$$H(W^* \mathrm{SO}(n)) = \mathrm{Span}\{\textit{Pontryagin \& Secondary Classes}\}.$$

It follows that we get a characteristic map,

$$\hat{\theta} : H(W^* \mathrm{SO}(n)) \xrightarrow{\sim} HC^*(\mathcal{H}_n) \xrightarrow{\theta} HC^*(C_c^\infty(P) \rtimes \Gamma).$$

Proposition (Connes-Moscovici '98)

The class of the CM cocycle is contained in the range of the characteristic map.

Theorem (Connes-Moscovici '98)

There is a universal class $L_n \in H(W^* \, \mathrm{SO}(n))$ *such that, for any oriented manifold* M^n *and any group* Γ *of orientation-preserving diffeomorphisms of* M*, we have*

$$\mathrm{ind}_D[\mathcal{E}] = \langle \hat{\theta}(L_n), \mathcal{E} \rangle \qquad \forall \mathcal{E} \in K_0(C_c^\infty(P) \rtimes \Gamma).$$