# Noncommutative Geometry
# Lecture 4: The Local Index Formula in Noncommutative Geometry

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# The Connes-Chern Character

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$$\mu_k(D^{-1}) = \mathrm{O}(k^{-\frac{1}{p}}).$$

That is, $D^{-1}$ is an infinitesimal operator of order $1/p$.

## Lemma

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$$\operatorname{Tr}\left[\left|[F, a^1] \cdots [F, a^q]\right|\right] < \infty \qquad \forall a^j \in \mathcal{A}.$$

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**Definition (Connes)**

For $n > \frac{1}{2}(p+1)$ let $\tau_{2n}$ be the $2n$-cochain defined by

$$\tau_{2n}(a^0, \cdots, a^{2n}) = \frac{1}{2}\frac{n!}{(2n)!}\,\mathrm{Tr}\left[\gamma F[F, a^0] \cdots [F, a^{2n}]\right], \quad a^j \in \mathcal{A}.$$

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The class of $\tau_{2n}$ in $HC^{\text{even}}(\mathcal{A})$ is called the *Connes-Chern character* and is denoted $\mathrm{Ch}(\mathcal{A}, D)$.

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**Theorem (Connes)**

*For all $\mathcal{E} \in K_0(\mathcal{A})$,*

$$\mathrm{ind}_D[\mathcal{E}] = \langle \mathrm{Ch}(\mathcal{A}, D), \mathcal{E} \rangle.$$

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1. The operator $F$ which is like a $\psi$DO.
2. The operator trace which is not a local functional, i.e., it does not vanish on infinitesimals of a given order.

Therefore, it was sought for a more convenient representative of the Connes-Chern character.

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**Remark**

$$p\text{-summability} \implies \theta\text{-summability}.$$

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where $\Delta_{2k}$ is the $2k$-simplex

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**Remark**

As observed by Quillen, $\varphi^t_{\mathrm{JLO}}$ can be interpreted as the Chern character of a superconnection on the algebra of cochains.

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**Remark**

1. $\varphi^t_{2k} \neq 0$ for large $k$, so $\varphi^t_{\text{JLO}}$ is NOT a cochain in $C^{\text{even}}(\mathcal{A})$.
2. This is a cocycle in *entire cyclic cohomology*, i.e., in the cohomology of infinite cochains $\varphi = (\varphi_0, \varphi_2, \cdots)$ such that, for any finite subset $S \subset \mathcal{A}$, the power series,

$$\sum_{k \geq 0} \frac{z^k}{k!} \varphi_{2k}(a^0, \cdots, a^{2k}), \qquad a^j \in S,$$

are entire functions.

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**Theorem (Connes)**

*Connes's cocycle $\tau_{2n}^{D}$ and the JLO cocycle $\varphi_{\mathrm{JLO}}^{t}$ are cohomologous in entire cyclic cohomology.*

### Assumption

# Retraction of the JLO Cocycle

**Assumption**

As $t \to 0^+$,

$$\varphi_{2k}^{t} = \sum_{\substack{\alpha, l \geq 0 \\ \alpha + l > 0}} t^{-\alpha} (\log^l t) \varphi_{2k}^{(\alpha, l)} + \varphi_{2k}^{(0,0)} + \mathrm{o}(t),$$

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**Definition**

The finite part of the JLO cocycle is

$$\mathrm{FP}_{t \to 0^+} \varphi_{\mathrm{JLO}}^t := \left( \varphi_0^{(0,0)}, \varphi_2^{(0,0)}, \cdots \right).$$

Theorem (Connes-Moscovici)

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$$\mathrm{FP}_{t\to 0^+}\,\varphi^t_{\mathrm{JLO}} = (\varphi_0, \varphi_{2k}, \cdots),$$

$$\varphi_{2k}(f^0, \cdots, f^{2k}) = \frac{(2i\pi)^{-\frac{n}{2}}}{(2k)!}\int_M f^0 df^1 \wedge \cdots \wedge df^k \wedge \hat{A}(R^M).$$

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For $T \in \mathcal{L}(\mathcal{H})$ set

$$\delta^0(T) = T, \quad \delta^1(T) := [|D|, T], \qquad \delta^2(T) := [|D|, [|D|, T]],$$
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**Definition**

$\mathcal{B}$ is the algebra generated by $\gamma$ and the $\delta^j(a)$ and $\delta^j([D, a])$, $a \in \mathcal{A}$.

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## Definition

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## Fact

*For any $b \in \mathcal{B}$, the function $\zeta_b(z) := \operatorname{Tr}\left[b|D|^{-z}\right]$ is analytic for $\Re z \gg 1$.*

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**Example**

A Dirac spectral triple $\left(C^\infty(M), L^2(M, \not{S}), \not{D}\right)$ has dimension spectrum,

$$\Sigma = \{k \in \mathbb{Z};\ k \leq \dim M\}\,.$$

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$\Psi_D^q(\mathcal{A})$, $q \in \mathbb{C}$, is the space of operators such that

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where $\simeq$ means that, for all $N \in \mathbb{N}$ and $s \in \mathbb{R}$,

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*where the sum is finite, the* $c_{k,\alpha}$ *are universal constants, and we have set*

$$T^{[j]} := \overbrace{[D^2, [D^2, \cdots [D^2, T] \cdots ]]}^{j \ \mathit{times}}.$$

2. *The CM cocycle represents the Connes-Chern character, and hence*

$$\mathrm{ind}_D[\mathcal{E}] = \langle \varphi_{\mathrm{CM}}, \mathcal{E} \rangle \qquad \forall \mathcal{E} \in K_0(\mathcal{A}).$$

# The CM Cocycle

## Example

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For a Dirac spectral triple $\left(C^\infty(M), L^2(M, \not{S}), \not{D}\right)$,

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For a Dirac spectral triple $\left(C^\infty(M), L^2(M, \not{S}), \not{D}\right)$,

$$\varphi_{\mathrm{CM}} = (\varphi_0, \varphi_{2k}, \cdots),$$

$$\varphi_{2k}(f^0, \cdots, f^{2k}) = \frac{(2i\pi)^{-\frac{n}{2}}}{(2k)!} \int_M f^0 df^1 \wedge \cdots \wedge df^{2k} \wedge \hat{A}(R^M).$$