Noncommutative Geometry
Lecture 3: Cyclic Cohomology

Raphaël Ponge

Seoul National University

October 25, 2011

Hochschild Cohomology

Setup

$\mathcal{A}$ is a unital algebra over $\mathbb{C}$.

Definition (Hochschild Complex)

1. The space of *n-cochains*, $n \geq 0$, is

$$C^n(\mathcal{A}) := \left\{(n+1)\text{-linear forms } \varphi : \mathcal{A}^{n+1} \to \mathbb{C}\right\}, \quad n \geq 0.$$

2. The *coboundary* $b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ is given by

$$(b\varphi)(a^0, \cdots, a^{n+1}) = \sum_{0 \leq j \leq n} (-1)^j \varphi(a^0, \cdots, a^j a^{j+1}, \cdots, a^{n+1}) + (-1)^{n+1}\varphi(a^{n+1}a^0, \cdots, a^n).$$

Hochschild Cohomology

Lemma

We have $b^2 = 0$.

Definition

The cohomology of the complex $(C^\bullet(\mathcal{A}), b)$ is called the Hoschschild cohomology of $\mathcal{A}$ and is denoted $HH^\bullet(\mathcal{A})$.

Example

Let C be a k-dimensional current on a compact manifold M. Define a k-cochain on $\mathcal{A} = C^\infty(M)$ by

$$\varphi_C(f^0, f^1, \cdots, f^k) = \frac{1}{k!}\langle C, f^0 df^1 \wedge \cdots \wedge df^k \rangle.$$

Then $b\varphi_C = 0$. In fact, we have

Theorem (Hochschild-Kostant-Rosenberg, Connes)

There is an isomorphism,

$$HH^k(M) \simeq \mathcal{D}'_k(M).$$

Cyclic Cohomology (Connes, Tsygan)

Definition (Cyclic Cochains)

A cochain $\varphi \in C^n(\mathcal{A})$, $n \geq 0$, is *cyclic* when

$$\varphi(a^1, \cdots, a^n, a^0) = (-1)^n \varphi(a^0, \cdots, a^n) \quad \forall a^j \in \mathcal{A}.$$

We denote by $C^n_\lambda(\mathcal{A})$ the space of cyclic n-cochains.

Example

Let C be a k-dimensional current on a compact manifold M. We saw it defines a Hochschild cocycle, Then

$$C \text{ closed (i.e., } d^t C = 0) \implies \varphi_C \text{ cyclic.}$$

Cyclic Cohomology

Lemma

$$\varphi \text{ cyclic} \implies b\varphi \text{ cyclic.}$$

Definition

The cohomology of the sub-complex $(C^\bullet_\lambda(\mathcal{A}), b)$ is called the *cyclic cohomology* of $\mathcal{A}$ and is denoted $HC^\bullet(\mathcal{A})$.

Periodic Cyclic Cohomology (Connes, Tsygan)

Definition

Define $B : C^n(\mathcal{A}) \to C^{n-1}(\mathcal{A})$ by

$$B = AB_0,$$

where $B_0 : C^n(\mathcal{A}) \to C^{n-1}(\mathcal{A})$ and $A : C^{n-1}(\mathcal{A}) \to C^{n-1}(\mathcal{A})$ are given by

$$B_0\varphi(a^0, \cdots, a^{n-1}) = \varphi(1, a^0, \cdots, a^{n-1}) - (-1)^n \varphi(a^0, \cdots, a^{n-1}, 1),$$

$$A\psi(a^0, \cdots, a^{n-1}) = \sum (-1)^{j(n-1)} \psi(a^j, \cdots, a^{n-1}, a^0, \cdots, a^{j-1}).$$

Remark

1. If φ is a cyclic cochain, then $B_0\varphi = 0$, and hence $B\varphi = 0$.
2. An n-cochain φ is cyclic if and only if $A\varphi = \frac{1}{n+1}\varphi$.

Example

Let C be a k-dimensional current on a compact manifold M. It defines a Hochschild cocycle,

$$\varphi_C(f^0, f^1, \cdots, f^k) = \frac{1}{k!}\langle C, f^0 df^1 \wedge \cdots \wedge df^k \rangle.$$

We then have

$$B\varphi_C = \varphi_{d^tC}.$$

Lemma

We have

$$B^2 = 0 \qquad \textit{and} \qquad bB + Bb = 0.$$

Periodic Cyclic Cohomology

Definition (Even/Odd Cochains)

Define

$$C^{\text{even}}(\mathcal{A}) := \bigoplus_{k\geq 0} C^{2k}(\mathcal{A})$$
$$= \left\{ \varphi = (\varphi_0, \varphi_2, \cdots);\ \varphi_{2k} \in C^{2k}(\mathcal{A}),\ \varphi_{2k} = 0 \text{ for large } k \right\},$$

and

$$C^{\text{odd}}(\mathcal{A}) := \bigoplus_{k\geq 0} C^{2k+1}(\mathcal{A})$$
$$= \left\{ \varphi = (\varphi_1, \varphi_3, \cdots);\ \varphi_{2k+1} \in C^{2k+1}(\mathcal{A}),\ \varphi_{2k+1} = 0 \text{ for large } k \right\}.$$

Periodic Cyclic Cohomology

Proposition

We have a 2-periodic complex,

$$C^{\text{even}}(\mathcal{A}) \overset{b+B}{\leftrightarrows} C^{\text{odd}}(\mathcal{A}).$$

Definition

The cohomology of $\left(C^{\text{even/odd}}(\mathcal{A}), b+B\right)$ is called the *periodic cyclic cohomology* of $\mathcal{A}$ and is denoted $HC^{\text{even/odd}}(\mathcal{A})$.

Example

Let $C = C_0 + C_2 + \cdots$ be an even current on a compact manifold M. Then C defines an even cochain,

$$\varphi_C = (\varphi_{C_0}, \varphi_{C_2}, \cdots),$$

$$\varphi_{C_{2k}}(f^0, f^1, \cdots, f^{2k}) = \frac{1}{(2k)!}\langle C_{2k}, f^0 df^1 \wedge \cdots \wedge df^{2k}\rangle.$$

Then

$$(b + B)\varphi_C = B\varphi_C = \varphi_{d^t C}.$$

$$C \text{ closed } \Longrightarrow \varphi_C \text{ even cyclic cocycle.}$$

Theorem (Connes)

The map $C \to \varphi_C$ gives rise to isomorphisms,

$$H_{\text{even/odd}}(M) \simeq HC^{\text{even/odd}}(C^\infty(M)).$$

Remark

Assume M is oriented, Riemannian and has even dimension. Then the $\hat{A}$-form $\hat{A}(R^M)$ defines an even cyclic cocycle by

$$\varphi_{\hat{A}(R^M)} = \varphi_{\hat{A}(R^M)^\vee},$$

i.e., $\varphi_{\hat{A}(R^M)} = (\varphi_0, \varphi_2, \cdots)$, with

$$\varphi_{2k}(f^0, f^1, \cdots, f^{2k}) = \frac{1}{(2k)!} \int_M f^0 df^1 \wedge \cdots \wedge df^{2k} \wedge \hat{A}(R^M).$$

Likewise the Pfaffian $\mathrm{Pf}(R^M)$, the L-form $L(R^M)$ and the Todd form $\mathrm{Td}(R^M)$ define even cyclic cocycles.

Morita Equivalence

Definition

Let $\varphi \in C^n(\mathcal{A})$. The n-cochain $\varphi \# \operatorname{Tr}$ on $M_q(\mathcal{A}) = \mathcal{A} \otimes M_q(\mathbb{C})$ is defined by

$$\varphi \# \operatorname{Tr}(a^0 \otimes \mu^0, \cdots, a^n \otimes \mu^n) := \varphi(a^0, \cdots, a^n) \operatorname{Tr}\left[\mu^0 \mu^1 \cdots \mu^n\right]$$

for all $a^j \in \mathcal{A}$ and $\mu^j \in M_q(\mathbb{C})$.

Lemma

We have

$$b\left(\varphi \# \operatorname{Tr}\right) = \left(b\varphi\right) \# \operatorname{Tr}.$$

Theorem (Connes)

The map $\varphi \to \varphi \# \operatorname{Tr}$ gives rise to isomorphisms,

$$HC^\bullet(\mathcal{A}) \simeq HC^\bullet\left(M_q(\mathcal{A})\right).$$

Morita Equivalence

Example

Let C be a k-dimensional current on a compact manifold M. For the cochain,

$$\varphi_C(f^0, f^1, \cdots, f^k) = \frac{1}{k!}\langle C, f^0 df^1 \wedge \cdots \wedge df^k\rangle,$$

we have

$$\varphi_C \# \operatorname{Tr}(a^0, a^1, \cdots, a^k) = \frac{1}{k!}\langle C, \operatorname{Tr}\left[a^0 da^1 \wedge \cdots \wedge da^k\right]\rangle$$

for all a^j in $M_q\left(C^\infty(M)\right) = C^\infty\left(M, M_q(\mathbb{C})\right)$.

Pairing with K-Theory ($\mathcal{A} = C^\infty(M)$)

Setup

- M is a compact manifold.
- $E = \operatorname{ran} e$, $e = e^2 \in M_q(C^\infty(M))$.
- F^E is the curvature of the Grassmanian connection ∇^E of E.

Lemma

1. $F^E = e(de)^2 = e(de)^2 e.$
2. $\mathrm{Ch}(F^E) = \sum \frac{(-1)^k}{k!} \operatorname{Tr}\left[e(de)^k\right].$

Pairing with K-Theory ($\mathcal{A} = C^\infty(M)$)

Proposition

Let $C = C_0 + C_2 + \cdots$ be a closed even current on M with associated even cocycle $\varphi_C = (\varphi_{C_0}, \varphi_{C_2}, \cdots)$. Then

$$\begin{aligned}\langle C, E\rangle &= \sum (-1)^k \frac{(2k)!}{k!} \left(\varphi_{C_{2k}} \# \operatorname{Tr}\right)(e, e, \cdots e), \\ &= \varphi_{C_0}(e) + \sum_{k\geq 1} (-1)^k \frac{(2k)!}{k!} \left(\varphi_{C_{2k}} \# \operatorname{Tr}\right) \left(e - \frac{1}{2}, e, \cdots e\right).\end{aligned}$$

Pairing with K-Theory (General Case)

Setup

$\mathcal{A}$ is a unital algebra over $\mathbb{C}$.

Definition

A cochain $\varphi \in C^n(\mathcal{A})$, $n \geq 1$, is *normalized* when

$$\varphi(a^0, a^1, \cdots, a^n) = 0 \quad \text{whenever } a^j = 1 \text{ for some } j \geq 1.$$

Lemma

Any class in $HC^{\mathrm{even}}(\mathcal{A})$ contains a normalized representative.

Pairing with K-Theory

Example

Let C be a k-dimensional current on a compact manifold M with associated cochain,

$$\varphi_C(f^0, f^1, \cdots, f^k) = \frac{1}{k!}\langle C, f^0 df^1 \wedge \cdots \wedge df^k \rangle.$$

Then φ_C is a normalized cochain.

Pairing with K-Theory

Definition

Let $\varphi = (\varphi_0, \varphi_2, \cdots)$ be an even cyclic cocycle and let $\mathcal{E} = e\mathcal{A}^q$, $e = e^2 \in M_q(\mathcal{A})$, a finitely generated projective module. The pairing of φ and $\mathcal{E}$ is

$$\langle \varphi, \mathcal{E} \rangle := \varphi_0(e) + \sum_{k \geq 1} (-1)^k \frac{(2k)!}{k!} (\varphi_{2k} \# \mathrm{Tr}) \left(e - \frac{1}{2}, e, \cdots e \right).$$

Theorem (Connes)

The above pairing descends to a bilinear pairing,

$$\langle \cdot, \cdot \rangle : HC^{\mathrm{even}}(\mathcal{A}) \times K_0(\mathcal{A}) \longrightarrow \mathbb{C}.$$

Example

Let $C = C_0 + C_2 + \cdots$ be a closed even current on a compact manifold M and let $E = \operatorname{ran} e$, $e = e^2 \in M_q(C^\infty(M))$, so that $\mathcal{E} = C^\infty(M, E) \simeq C^\infty(M)^q$. Then

$$\langle \varphi_C, \mathcal{E} \rangle = \langle C, E \rangle.$$

Example

Assume M is spin, oriented, Riemannian and has even dimension.

1. For $C = \hat{A}(R^M)^\vee$ and the Dirac operator,
$$\langle \varphi_{\hat{A}(R^M)}, \mathcal{E}\rangle = \langle \hat{A}(R^M)^\vee, E\rangle \quad \text{and} \quad \operatorname{ind}_{\not{D}}[\mathcal{E}] = \operatorname{ind}_{\not{D}}[E].$$
2. By the K-theoretic version of the Atiyah-Singer Index Theorem explained in Lecture 2,
$$\operatorname{ind}_{\not{D}}[E] = (2i\pi)^{-\frac{n}{2}}\langle \hat{A}(R^M)^\vee, E\rangle.$$
3. Therefore, the Atiyah-Singer Index Theorem can be further restated as

Theorem

$$\operatorname{ind}_{\not{D}}[\mathcal{E}] = (2i\pi)^{-\frac{n}{2}}\langle \varphi_{\hat{A}(R^M)}, \mathcal{E}\rangle \quad \forall \mathcal{E} \in K_0(C^\infty(M)).$$