Noncommutative Geometry
Lecture 1: Quantized Calculus

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Quantum Mechanics vs. General Relativity

**Fundamental Problem**
- Unify general relativity and quantum mechanics.
- Find a common mathematical framework for general relativity and quantum mechanics.

**NCG Approach**
Translate the tools of Riemannian geometry into the Hilbert space formalism of quantum mechanics.
Diffeomorphism Invariant Geometry

Setup

- $M$ smooth manifold.
- $\Gamma$ is a group of diffeomorphisms acting on $M$.

Remark

1. If $\Gamma$ acts freely and properly, then $M/\Gamma$ is a smooth manifold.
2. In general, $M/\Gamma$ need not be Hausdorff!!!
Observation

The algebra $C^\infty_c(M/\Gamma)$ always makes sense when realized as the crossed-product algebra,

$$C^\infty_c(M) \rtimes \Gamma := \left\{ \text{finite sums} \sum_{\varphi \in \Gamma} f_\varphi U_\varphi; \ f_\varphi \in C^\infty_c(M) \right\},$$

where the $f_\varphi$ and $U_\varphi$ are represented as operators such that

$$U_\varphi^* = U_\varphi^{-1} = U_{\varphi^{-1}}, \quad U_\varphi f = (f \circ \varphi^{-1}) U_\varphi.$$

Theorem (Green)

*If $\Gamma$ acts freely and properly, then $C^\infty_c(M/\Gamma) \simeq C^\infty_c(M) \rtimes \Gamma$.***
The Noncommutative Torus

Example

Given \( \theta \in \mathbb{R} \), let \( \mathbb{Z} \) act on \( S^1 \) by

\[
k.z := e^{2ik\pi \theta} z \quad \forall z \in S^1 \quad \forall k \in \mathbb{Z}.
\]

Remark

If \( \theta \notin \mathbb{Q} \), then the orbits of the action are dense in \( S^1 \).

The crossed-product algebra \( A_\theta := C^\infty(S^1) \rtimes_\theta \mathbb{Z} \) is generated by two operators \( U \) and \( V \) such that

\[
U^* = U^{-1}, \quad V^* = V^{-1}, \quad UV = e^{2i\pi \theta} UV.
\]

Remark

The algebra \( A_\theta \) is called the noncommutative torus.
Gel’fand Transform

Theorem (Gel’fand-Naimark)

Any $C^*$-algebra can be realized as a closed self-adjoint subalgebra of some $\mathcal{L}(\mathcal{H})$.

Theorem (Gel’fand-Naimark)

There is a one-to-one correspondence,

\[ \{ \text{Locally Compact Spaces} \} \quad \longleftrightarrow \quad \{ \text{Commutative } C^*\text{-algebras} \} \]

\[ X \quad \longrightarrow \quad C_0(X) \subset \mathcal{L}(L^2(X)) \]
<table>
<thead>
<tr>
<th>Classical Calculus</th>
<th>Quantized Calculus</th>
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</thead>
<tbody>
<tr>
<td>Complex Variable</td>
<td>Operator on $\mathcal{H}$</td>
</tr>
<tr>
<td>Real Variable</td>
<td>Selfadjoint Operator on $\mathcal{H}$</td>
</tr>
<tr>
<td>Infinitesimal Variable</td>
<td>Compact Operator</td>
</tr>
<tr>
<td>Infinitesimal of Order $\alpha$</td>
<td>Compact Operator $T$</td>
</tr>
<tr>
<td>Differential $df = \sum \frac{\partial f}{\partial x^\mu} dx^\mu$</td>
<td>such that $\mu_n(T) = O(n^{-\alpha})$</td>
</tr>
<tr>
<td>Integral $\int f$</td>
<td>Quantized Differential $da = [F, a]$</td>
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<td>Dixmier Trace $\int T$</td>
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The Atiyah-Singer Index Theorem

**Definition**

The *Fredholm index* of $\mathcal{D}_E$ is

$$\text{ind} \mathcal{D}_E := \dim \ker \left[ (\mathcal{D}_E)_{\mathbb{S}^+ \otimes E} \right] - \dim \ker \left[ (\mathcal{D}_E)_{\mathbb{S}^- \otimes E} \right].$$

**Theorem (Atiyah-Singer)**

$$\text{ind} \mathcal{D}_E = (2i\pi)^{-\frac{n}{2}} \int_M \hat{A}(R^M) \wedge \text{Ch}(F^E),$$

where:

- $\hat{A}(R^M) := \det \frac{1}{2} \left[ \frac{R^M/2}{\sinh(R^M/2)} \right]$ is called the $\hat{A}$-class of the curvature $R^M$ of $M$.
- $\text{Ch}(F^E) := \text{Tr} \left[ e^{-F^E} \right]$ is called the Chern form of the curvature $F^E$ of $\nabla^E$. 