Noncommutative Geometry
Lecture 1: Quantized Calculus

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Quantum Mechanics vs. General Relativity

Fundamental Problem

- Unify general relativity and quantum mechanics.
- Find a a common mathematical framework for general relativity and quantum mechanics.

NCG Approach

Translate the tools of Riemannian geometry into the Hilbert space formalism of quantum mechanics.

Diffeomorphism Invariant Geometry

Setup

- M smooth manifold.
- Γ is a group of diffeomorphisms acting on M.

Remark

1. If Γ acts freely and properly, then M/Γ is a smooth manifold.
2. In general, M/Γ need not be Hausdorff!!!

Crossed-Product Algebra $C_c^\infty(M) \rtimes \Gamma$

Observation

The algebra $C_c^\infty(M/\Gamma)$ always makes sense when realized as the crossed-product algebra,

$$C_c^\infty(M) \rtimes \Gamma := \left\{ \text{finite sums} \sum_{\varphi \in \Gamma} f_\varphi U_\varphi ;\ f_\varphi \in C_c^\infty(M) \right\},$$

where the f_φ and U_φ are represented as operators such that

$$U_\varphi^* = U_\varphi^{-1} = U_{\varphi^{-1}}, \qquad U_\varphi f = (f \circ \varphi^{-1}) U_\varphi.$$

Theorem (Green)

If Γ acts freely and properly, then $C_c^\infty(M/\Gamma) \simeq C_c^\infty(M) \rtimes \Gamma$.

The Noncommutative Torus

Example

Given $\theta \in \mathbb{R}$, let $\mathbb{Z}$ act on S^1 by

$$k.z := e^{2ik\pi\theta} z \qquad \forall z \in S^1 \; \forall k \in \mathbb{Z}.$$

Remark

If $\theta \notin \mathbb{Q}$, then the orbits of the action are dense in S^1.

The crossed-product algebra $\mathcal{A}_\theta := C^\infty(S^1) \rtimes_\theta \mathbb{Z}$ is generated by two operators U and V such that

$$U^* = U^{-1}, \qquad V^* = V^{-1}, \qquad VU = e^{2i\pi\theta} UV.$$

Remark

The algebra $\mathcal{A}_\theta$ is called the *noncommutative torus*.

Gel'fand Transform

Theorem (Gel'fand-Naimark)

Any C^*-algebra can be realized as a closed self-adjoint subalgebra of some $\mathcal{L}(\mathcal{H})$.

Theorem (Gel'fand-Naimark)

There is a one-to-one correspondence,

$$\begin{array}{ccc} \{\text{Locally Compact Spaces}\} & \longleftrightarrow & \{\text{Commutative } C^*\text{-algebras}\} \\ X & \longrightarrow & C_0(X) \subset \mathcal{L}\left(L^2(X)\right) \end{array}.$$

Quantized Calculus

Classical Calculus	Quantized Calculus
Complex Variable	Operator on $\mathcal{H}$
Real Variable	Selfadjoint Operator on $\mathcal{H}$
Infinitesimal Variable	Compact Operator
Infinitesimal of Order α	Compact Operator T such that $\mu_n(T) = O(n^{-\alpha})$
Differential $df = \sum \frac{\partial f}{\partial x^\mu} dx^\mu$	Quantized Differential $da = [F, a]$
Integral $\int f$	Dixmier Trace $⨍ T$

The Atiyah-Singer Index Theorem

Definition

The *Fredholm index* of $\not{D}_E$ is

$$\operatorname{ind} \not{D}_E := \dim \ker \left[(\not{D}_E)_{|\not{S}^+ \otimes E} \right] - \dim \ker \left[(\not{D}_E)_{|\not{S}^- \otimes E} \right].$$

Theorem (Atiyah-Singer)

$$\operatorname{ind} \not{D}_E = (2i\pi)^{-\frac{n}{2}} \int_M \hat{A}(R^M) \wedge \mathrm{Ch}(F^E),$$

where:

- $\hat{A}(R^M) := \det^{\frac{1}{2}} \left[\frac{R^M/2}{\sinh\left(R^M/2\right)} \right]$ *is called the $\hat{A}$-class of the curvature R^M of M.*
- $\mathrm{Ch}(F^E) := \mathrm{Tr}\left[e^{-F^E} \right]$ *is called the Chern form of the curvature F^E of ∇^E.*