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Sujet : **Calcul hypoelliptique sur les variétés de Heisenberg, résidu non commutatif et géométrie pseudo-hermitienne**

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HAD I the heavens' embroidered cloths,
Enwrought with golden and silver light,
The blue and the dim and the dark cloths
Of night and light and the half-light,
I would spread the cloths under your feet:
But I, being poor, have only my dreams;
I have spread my dreams under your feet;
Tread softly because you tread on my dreams.

W.B. Yeats *He Wishes For The Cloths Of Heaven*
in *The Wind Among The Reeds.*

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Contents

Version française abrégée

Dans cette thèse on s'attache à démontrer divers théorèmes en géométrie pseudo-hermitienne, et plus généralement pour les variétés de Heisenberg, comme applications de la construction d'un résidu non-commutatif dans le cadre du calcul hypoelliptique sur les variétés de Heisenberg.

La thèse est organisée comme suit. Dans le premier chapitre on fait une revue complète sur le calcul pseudo-différentiel sur les variétés de Heisenberg, qu'on appellera pour simplifier $\Psi_{\mathcal{V}}DO$-calcul, tel qu'il est présenté dans [BG] et [BGS].

Dans le second chapitre on développe un $\Psi_{\mathcal{V}}DO$-calcul avec paramètre permettant une construction pseudo-différentielle de la résolvante d'un sous-laplacien elliptique (théorème 2.17).

Au chapitre 3 on définit et étudie les familles holomorphes de $\Psi_{\mathcal{V}}DO$ qu'on utilise pour construire les puissances complexes d'un sous-laplacien elliptique (théorèmes 3.6, 3.10 et 3.11).

Dans le chapitre 4 on construit un prolongement analytique de la trace pour les $\Psi_{\mathcal{V}}DO$ d'ordre complexe non entier et on montre qu'aux $\Psi_{\mathcal{V}}DO$ d'ordre entier on a une trace résiduelle qui l'analogue complet du résidu non commutatif (théorème 4.4 et proposition 4.6). On montre ensuite que ce résidu non commutatif permet d'étendre la trace de Dixmier à toute l'algèbre des $\Psi_{\mathcal{V}}DO$ d'ordre entier (théorème 4.7) et que c'est essentiellement l'unique trace sur cette algèbre modulo les opérateurs régularisants (théorème 4.8).

Dans le dernier chapitre on donne des applications géométriques du résidu non commutatif et de la trace régularisée. D'abord on définit la fonction zêta d'un sous-laplacien elliptique dans le $\Psi_{\mathcal{V}}DO$-calcul et on relie ses résidus et valeurs régulières aux coefficients du développement de la chaleur (théorèmes 5.2 et 5.4). On obtient ensuite des formules variationelles pour les fonctions zêta qu'on utilise pour produire des invariants conformes d'une variété pseudo-hermitienne (théorème 5.8). Après on étudie la géométrie non-commutative des variétés de Heisenberg. En particulier on définit l'aire d'une variété pseudo-hermitienne et on montre qu'en dimension 3 cette aire est donnée par une formule locale invoquant la courbure scalaire de Tanaka-Webster (théorème 5.13).

Enfin dans la dernière section on donne des formules locales pour calculer l'indice d'une racine carrée d'un sous-laplacien elliptique. D'abord on montre qu'en dimension paire l'indice est toujours égal à zéro et qu'en dimension impaire il est donné par l'intégrale de la densité qui apparait comme le terme constant dans l'asymptotique du noyau de la chaleur du sous-laplacien (théorème 5.14). Ensuite, en utilisant la cohomologie cyclique et la formule d'indice locale de Connes-Moscovici [CM2], on montre qu'il existe un courant de Rham, calculable par des formules locales explicites, dont l'accouplement avec le caractère de Chern donne l'indice à coefficients dans la K^0-théorie de la variété (théorème 5.15).

Chapitre 1 : calcul hypoelliptique sur les variétés de Heisenberg

Dans ce chapitre on fait une revue du $\Psi_{\mathcal{V}}DO$-calcul, aussi appelé $\Psi_{\mathcal{V}}DO$-calcul, tel qu'il est présenté dans [BG] et [BGS].

Variétés de Heisenberg

Une *variété de Heisenberg* $(M, \mathcal{V})$ est une variété M avec un fibré en hyperplans $\mathcal{V} \subset TM$. Un difféomorphisme $\phi : (M, \mathcal{V}) \to (M', \mathcal{V}')$ entre deux variétés de Heisenberg est dit Heisenberg quand $\phi_* \mathcal{V} = \mathcal{V}'$.

Le modèle local d'une variété de Heisenberg de dimension $(d+1)$ est un ouvert U de $\mathbb{R}^{d+1}$ avec un fibré en hyperplans $\mathcal{V} \subset TU$ et est un $\mathcal{V}$-*repère* $X_0, X_1, \ldots, X_d$ de TU, i.e. $X_0, X_1, \ldots, X_d$ est un repère de TU et $X_1, \ldots, X_d$ engendrent $\mathcal{V}$. On définit alors une carte Heisenberg (locale) pour une variété de Heisenberg comme un difféomorphisme Heisenberg (local) vers un tel ouvert.

On a les exemples suivants de variétés de Heisenberg : groupe de Heisenberg, feuilletages (de codimension 1), feuilletacts (confeuilletages) [ET], variétés de contact, CR, pseudo-hermitiennes.

La raison pour laquelle on utilise la terminologie *variété de Heisenberg* provient de ce qu'on a en chaque point de la variété un groupe tangent de la forme $H_{2n+1} \times \mathbb{R}^{d-2n}$. Par exemple dans le cas d'une variété de contact M^{2n+1} on obtient le groupe de Heisenberg H_{2n+1} en chaque point, tandis qu'à l'opposé pour un feuilletage de codimension 1 on obtient toujours le groupe abélien $\mathbb{R}^{d+1}$.

Sous-laplaciens et idées derrière le $\Psi_{\mathcal{V}}DO$-calcul

Soit $(M^{d+1}, \mathcal{V})$ une variété de Heisenberg. On appelle *sous-laplacien* un opérateur différentiel sur M qui localement est de la forme

$$\Delta = -\sum_{j=1}^{d} X_j^2 - i\lambda(x)X_0 + \sum_{j=1}^{d} \mu_j(x)X_j + \nu(x), \qquad (1)$$

où $\lambda, \mu_1, \ldots, \mu_d, \nu$ sont des fonctions lisses et $X_0, X_1, \ldots, X_d$ est un $\mathcal{V}$-repère local de TM.

Ce type d'opérateur ne peut être elliptique. Néanmoins il peut être hypoelliptique et dans ce cas le $\Psi_{\mathcal{V}}DO$-calcul permet de construire une paramétrix. L'idéé est d'abord de considérer

$$\Delta_2 = -\sum_{j=1}^{d} X_j^2 - i\lambda(x)X_0, \tag{2}$$

comme ayant ordre 2 dans le $\Psi_{\mathcal{V}}DO$-calcul grâce aux dilatations

$$\lambda.\xi = (\lambda^2\xi_0, \lambda\xi_1, \ldots, \lambda\xi_d), \qquad \lambda > 0, \tag{3}$$

puis de figer les coefficients de Δ_2 en le modélisant en chaque point y par un opérateur différentiel Δ_2^y invariant à gauche sur le groupe tangent en y. Sous une certaine condition sur la fonction λ on montre que Δ_2^y est inversible et que l'inverse fournit le symbole principal d'une paramétrix pour Δ.

Donnons, dans le cas d'une variété pseudo-hermitienne (M^n, θ), quelques exemples de sous-laplaciens :

1) le laplacien de Kohn $\Box_b$ qui agit sur les formes CR complexes (voir [Ko]);

2) le sous-laplacien pseudo-hermtien $\Delta_b = 2\Re\Box_b$ introduit par Lee [Le];

3) le sous-laplacien conforme $\boxdot_\theta = \Delta_b + \frac{n}{n+2}R_n$, où R_n est la courbure scalaire de la connexion de Tanaka-Webster ([JL1]).

Le $\Psi_{\mathcal{V}}DO$-calcul

À partir de maintenant U est un ouvert de $\mathbb{R}^{d+1}$ avec un fibré en hyperplans $\mathcal{V} \subset TU$ et un $\mathcal{V}$-repère $X_0, X_1, \ldots, X_d$ de TU. Pour $x \in U$ on note par ε_x l'unique changement de coordonnées affine qui envoie x sur l'origine et tel que pour tout j le champs de vecteurs X_j coïncide avec $\frac{\partial}{\partial_{x_j}}$ en l'origine. On appelle ces nouvelle coordonnées les x-coordonnées. On pose aussi

$$\sigma(x,\xi) = (\sigma_0(x,\xi), \sigma_1(x,\xi), \ldots, \sigma_d(x,\xi)), \tag{4}$$

où $\sigma_j(x,\xi)$ est le symbole de $\frac{1}{i}X_j$. On dit alors que σ le symbole (réel) du repère $X_0, X_1, \ldots, X_d$.

Les symboles dans le $\Psi_{\mathcal{V}}DO$-calcul sont associés aux dilatations (3) et à la pseudo-norme homogène

$$\|\xi\| = (|\xi_0|^2 + |\xi_1|^4 + \ldots + |\xi_d|^4)^{\frac{1}{4}}, \qquad \xi \in \mathbb{R}^{d+1}. \tag{5}$$

Définition 1.1 *$S_m(U \times \mathbb{R}^{d+1})$, $m \in \mathbb{C}$, est l'espace des fonctions $f \in C^\infty(U \times \mathbb{R}^{d+1}\backslash 0)$ qui sont homogènes de degré m en la dernière variable, i.e.*

$$f(x, \lambda.\xi) = \lambda^m f(x,\xi), \qquad \lambda > 0. \tag{6}$$

Définition 1.2 *$S^k_{||}(U\times\mathbb{R}^{d+1})$, $k\in\mathbb{R}$, est l'espace des fonctions $f\in C^\infty(U\times\mathbb{R}^{d+1})$ satisfaisant aux estimées suivantes*

$$|\partial_x^\alpha\partial_\xi^\beta f(\xi)|\leq C_{\alpha\beta}(x)(1+\|\xi\|)^{k-\langle\alpha\rangle}, \tag{7}$$

où $C_{\alpha\beta}(x)$ est localement bornée sur U et où on a posé $\langle\alpha\rangle=\alpha_0+|\alpha|=2\alpha_0+\alpha_1+\ldots+\alpha_d$.

Définition 1.3 *$S^m(U\times\mathbb{R}^{d+1})$, $m\in\mathbb{C}$, est l'espace des fonctions $f\in C^\infty(U\times\mathbb{R}^{d+1})$ avec un développement asymptotique*

$$f(x,\xi)\sim\sum_{j\geq 0}f_{m-j}(x,\xi),\qquad f_k\in S_k(U\times\mathbb{R}^{d+1}), \tag{8}$$

au sens où pour tout entier N on a

$$|\partial_x^\alpha\partial_\xi^\beta(f-\sum_{j<N}f_{m-j})(x,\xi)|\leq C_{\alpha\beta N}(x)\|\xi\|^{\Re m-\langle\alpha\rangle-N},\qquad \|\xi\|\geq 1. \tag{9}$$

avec $C_{\alpha\beta NJ}(x)$ localement bornée sur U.

Définition 1.4 *Un $\Psi_{\mathcal{V}}DO$ opérateur d'ordre m, $m\in\mathbb{C}$, est un opérateur continu de $C_c^\infty(U)$ vers $C^\infty(U)$ de la forme*

$$P=f(x,\sigma(x,D))+R, \tag{10}$$

avec $f\in S^m(U\times\mathbb{R}^{d+1})$, appelé le symbole de P, et R opérateur régularisant. L'espace des $\Psi_{\mathcal{V}}DO$ d'ordre m est noté $\Psi_{\mathcal{V}}^m(U)$.

Proposition 1.5 ([BG]) *Soit $m\in\mathbb{C}$. Alors:*

1) L'espace $\Psi_{\mathcal{V}}^m(U)$ ne dépend pas du choix du $\mathcal{V}$-repère $X_0,X_1,\ldots,X_d$.

2) Chaque $P\in\Psi_{\mathcal{V}}^m(U)$ a un noyau lisse en dehors de la diagonale et s'étend en un opérateur continu de $\mathcal{E}'(U)$ vers $\mathcal{D}'(U)$. Cet opérateur est régularisant si, et seulement si, le symbole de P est dans $S^{-\infty}(U\times\mathbb{R}^{d+1})$.

3) Posons $k=\Re m$ si $\Re m\geq 0$ et $k=\frac{1}{2}\Re m$ sinon. Alors on a

$$\Psi_{\mathcal{V}}^m(U)\subset\Psi^k_{\frac{1}{2},\frac{1}{2}}(U) \tag{11}$$

où $\Psi^k_{\frac{1}{2},\frac{1}{2}}(U)$ est l'espace des opérateurs pseudo-différentiels de type $(\frac{1}{2},\frac{1}{2})$ (voir [Ho1]).

Combinant cette dernière inclusion avec le théorème de Calderón-Vaillancourt ([CV], [Hw]) on obtient la régularité Sobolev pour les $\Psi_{\mathcal{V}}DO$.

Proposition 1.6 *Soit $P \in \Psi^m_{\mathcal{V}}(U)$ et posons $k = \Re m$ si $\Re m \geq 0$ et $k = \frac{1}{2}\Re m$ sinon. Alors pour tout réel s l'opérateur P s'étend en une application linéaire continue*

$$P : H^s_{\text{comp}}(U) \longrightarrow H^{s-k}_{\text{loc}}(U). \tag{12}$$

Comme $\Psi^*_{\mathcal{V}}(U)$ est contenu dans $\Psi_{\frac{1}{2},\frac{1}{2}}(U)$ le développement asymptotique classique

$$q_1 \# q_2(x,\xi) \sum \frac{1}{\alpha!} D_\xi q_1(x,\xi) \partial_x q_2(x,\xi), \tag{13}$$

pour le symbole du produit de deux opérateurs pseudo-différentiels n'a plus de sens. Cependant on peut montrer que le produit de deux $\Psi_{\mathcal{V}}DO$ est encore un $\Psi_{\mathcal{V}}DO$ et donner alors un développement asymptotique pour le symbole du produit. Mais au lieu d'invoquer le produit ponctuel des symboles la formule s'exprime en terme d'une convolution en variable de Fourier par rapport au groupe tangent en chaque point.

Pour $y \in U$ on note X^y_j le vecteur invariant à gauche sur le y-groupe qui coïncide avec $\frac{\partial}{\partial x_j}$ en l'origine. On note alors $\sigma^y(x,\xi)$ le symbole du repère $X^y_0, X^y_1, \ldots, X^y_d$.

Lemme 1.7 ([BG]) *Soit $y \in U$. Alors:*

1) Pour tout $f \in S^k_{||}(\mathbb{R}^{d+1})$ l'opérateur $f(\sigma^y(x,D))$ envoie $\mathcal{S}(\mathbb{R}^{d+1})$ vers lui-même.

2) Pour $f_1 \in S^{k_1}_{||}(\mathbb{R}^{d+1})$ et $f_2 \in S^{k_2}_{||}(\mathbb{R}^{d+1})$ on a

$$f_1(\sigma^y(x,D)) \circ f_2(\sigma^y(x,D)) = (f_1 *^y f_2)(\sigma^y(x,D)), \tag{14}$$

où $y \to *^y$ est une famille lisse applications bilinéaire continues de $S^{k_1}_{||}(\mathbb{R}^{d+1}) \times S^{k_2}_{||}(\mathbb{R}^{d+1})$ vers $S^{k_1+k_2}_{||}(\mathbb{R}^{d+1})$.

Lemme 1.8 *Pour tous $m_1, m_2 \in \mathbb{C}$ la convolution $*$ pour les symboles dans $S^*_{||}(U \times \mathbb{R}^{d+1})$ induit une application bilinéaire*

$$* : S_{m_1}(U \times \mathbb{R}^{d+1}) \times S_{m_2}(U \times \mathbb{R}^{d+1}) \longrightarrow S_{m_1+m_2}(U \times \mathbb{R}^{d+1}). \tag{15}$$

Si $f(x,\xi)$ est un symbole on pose

$$f^{\beta\gamma}_\alpha(x,\xi) = \xi^\gamma \partial^\alpha_x \partial^\beta_\xi f(x,\xi) \qquad \text{and} \qquad f^\delta(x,\xi) = D^\delta_\xi f(x,\xi). \tag{16}$$

Ensuite on note $\sigma^{(x)}(z,\xi) = (\varepsilon_x)_* \sigma(z,\xi)$ le symbole du $\mathcal{V}$-repère dans les x-coordonnées et on pose

$$h_{\alpha\beta\gamma\delta}(x) = \frac{1}{\delta!}\partial^\delta_z h_{\alpha\beta\gamma}(x,0), \qquad h_{\alpha\beta\gamma}(x,z) = \frac{1}{\alpha!\beta!}(\varepsilon'^{-1}_x(z))^\alpha e_{\beta\gamma}(x,z), \tag{17}$$

où les fonctions $e_{\beta\gamma}$ sont définies par l'égalité

$$(\sigma^{(x)}(z,\xi) - \sigma^x(z,\xi))^\beta = \sum_{|\gamma|=|\beta|} e_{\beta\gamma}(x,z)\sigma^x(z,\xi)^\gamma. \tag{18}$$

Proposition 1.9 ([BG]) *Soit $P_1 \in \Psi_{\mathcal{V}}^{m_1}(U)$ de symbole $f_1 \sim \sum f_{1,m_1-j}$ et $P_2 \in \Psi_{\mathcal{V}}^{m_2}(U)$ de symbole $f_2 \sim \sum f_{2,m_2-j}$ et supposons que Q_1 ou Q_2 soit proprement supporté. Alors P_1P_2 est un $\Psi_{\mathcal{V}}DO$ d'ordre m_1+m_2 de symbole $f \sim \sum f_{m_1+m_2-j}$ avec*

$$f_{m_1+m_2-j}(x,\xi) = \sum h_{\alpha\beta\gamma\delta}(x) f_{1,m_1-k}^{\delta} * f_{2,m_2-l,\alpha}^{\beta\gamma}(x,\xi), \tag{19}$$

où la somme est prise sur les indices tels que

$$|\gamma| = |\beta|, \qquad |\beta| + |\alpha| \leq \langle\!\langle\delta\rangle\!\rangle + \langle\!\langle\beta\rangle\!\rangle - \langle\!\langle\gamma\rangle\!\rangle = j - k - l. \tag{20}$$

En particulier le symbole principal de P_1P_2 est $f_{1,m_1} * f_{2,m_2}$ la convolution des symboles principaux de P_1 et P_2.

On peut caractériser les $\Psi_{\mathcal{V}}DO$ par leurs noyaux de la façon suivante. Si $K(x)$ est une distribution tempérée sur $\mathbb{R}^{d+1}\lambda > 0$ on note $K(\lambda.x)$, $\lambda > 0$, la distribution tempérée définie par

$$\langle K(\lambda.x), u(x)\rangle = \lambda^{-(d+2)}\langle K(x), u(\lambda^{-1}.x)\rangle, \qquad u \in \mathcal{S}(\mathbb{R}^{d+1}). \tag{21}$$

Alors K est dite homogène de degré m, $m \in \mathbb{C}$, si

$$K(\lambda.x) = \lambda^m K(x) \qquad \text{pour tout } \lambda > 0. \tag{22}$$

Lemme 1.10 ([BG]) *Soit $f \in C^\infty(\mathbb{R}^{d+1}\backslash 0)$ homogène de degré m.*

1) Si m n'est pas un entier $\leq -(d+2)$, alors f s'étend de façon unique en une distribution tempérée homogène sur $\mathbb{R}^{d+1}$.

2) Si m est un entier $\leq -(d+2)$, les seules obstructions à une telle extension sont données la (les) non annulation(s) de

$$c_\alpha(f) = \frac{1}{\alpha!}\int_{\|\xi\|=1} \xi^\alpha f(\xi) i_E d\xi, \qquad \langle\!\langle\alpha\rangle\!\rangle = -(m+d+2), \tag{23}$$

où E est le générateur du flot $\phi_s(\xi) = (e^{2s}\xi_0, e^s\xi')$.

Cela amène à considérer les espaces suivants de distributions.

Définition 1.11 *$\mathcal{K}_m(U \times \mathbb{R}^{d+1})$, $m \in \mathbb{C}$, est l'espace des distributions $K(x,y) \in C^\infty(U)\hat{\otimes}\mathcal{S}'(\mathbb{R}^{d+1})$ qui sont lisses sur $U \times (\mathbb{R}^{d+1}\backslash 0)$ et telles que pour tout $\lambda > 0$ on ait*

$$K(x,\lambda.y) = \lambda^m K(x,y), \qquad \text{si } m \notin \mathbb{N}, \tag{24}$$

$$K(x,\lambda.y) = \lambda^m K(x,y) + \lambda^m \log\lambda \sum_{\langle\!\langle\alpha\rangle\!\rangle=m} c_\alpha(x) y^\alpha, \qquad \text{si } m \in \mathbb{N}. \tag{25}$$

Définition 1.12 *$\mathcal{K}^m(U \times \mathbb{R}^{d+1})$, $m \in \mathbb{C}$, est l'espace des distributions $K \in \mathcal{D}'(U \times \mathbb{R}^{d+1})$ admettant un développement asymptotique*

$$K(x,y) \sim \sum_{j \geq 0} K_{m+j}(x,y), \qquad K_l \in \mathcal{K}_l(U \times \mathbb{R}^{d+1}), \tag{26}$$

au sens où pour tout entier N si J est assez grand on ait

$$K - \sum_{j \leq J} K_{m+j} \in C^N(U \times \mathbb{R}^{d+1}) \tag{27}$$

Proposition 1.13 *Soit $K \in \mathcal{K}^m(U \times \mathbb{R}^{d+1})$. Alors:*

1) La distribution K appartient à $C^\infty(U)\hat{\otimes}\mathcal{D}'(\mathbb{R}^{d+1})$ et est C^∞ sur $U \times (\mathbb{R}^{d+1} \backslash 0)$.

2) Près de $y = 0$, si m est un entier ≥ 0 on a

(28)

$$\begin{aligned} K(x,y) &= \textstyle\sum_{0 \leq j \leq -\Re m} a_{m+j}(x,y) - c_K(x) \log \|y\| + \mathrm{O}(1), && \text{si } m \in \mathbb{N}, \\ K(x,y) &= \textstyle\sum_{0 \leq j \leq -\Re m} a_{m+j}(x,y) + \mathrm{O}(1), && \text{si } m \notin \mathbb{N}. \end{aligned}$$

3) Près de $y = 0$, si m est un entier ≥ 0 on a

$$K(x,y) = \sum_{0 \leq j \leq -\Re m} a_{m+j}(x,y) - c_K(x) \log \|y\| + \mathrm{O}(1), \tag{29}$$

et si m n'est pas un entier ≥ 0 on a

$$K(x,y) = \sum_{0 \leq j \leq -\Re m} a_{m+j}(x,y) + \mathrm{O}(1). \tag{30}$$

Dans les deux cas $a_k(x,y)$ est une fonction lisse sur $U \times (\mathbb{R}^{d+1} \backslash 0)$ homogène de degré k en la variable y.

Proposition 1.14 ([BG]) *Soit P un opérateur continu de $C_c^\infty(U)$ vers $C^\infty(U)$. Alors P est un $\Psi_\mathcal{V}DO$ d'ordre m si, et seulement si, son noyau est de la forme*

$$k_P(x,y) = |\varepsilon'_x| K(x, -\varepsilon_x(y)) + R(x,y), \tag{31}$$

avec $K \in \mathcal{K}^{\hat{m}}(U \times \mathbb{R}^{d+1})$, $\hat{m} = -(m+d+2)$, et $R \in C^\infty(U \times U)$.

Corollaire 1.15 *Soit P un $\Psi_\mathcal{V}DO$ d'ordre m entier. Alors près de la diagonale son noyau $k_P(x,y)$ a un comportement de la forme*

$$k_P(x,y) = \sum_{-(m+d+2) \leq j \leq 0} a_j(x, \varepsilon_x(y)) - c_P(x) \log \|\varepsilon_x(y)\| + \mathrm{O}(1), \tag{32}$$

avec $a_j(x,z)$ *homogène de degré* j *en la variable* z *et* $c_P(x)$ *donné par*

$$c_P(x) = \frac{|\varepsilon'_x|}{(2\pi)^{(d+2)}} \int_{\|\xi\|=1} f_{-(d+2)}(x,\xi) i_E d\xi, \tag{33}$$

où $f_{-(d+2)}$ *est le symbole (homogène) de degré* $-(d+2)$ *de* P*.*

Proposition 1.16 ([BG]) *Soit* $\phi : U \to \tilde{U}$ *un difféomorphisme Heisenberg, où* $\tilde{U}$ *est un autre ouvert de* $\mathbb{R}^{d+1}$ *avec un fibré en hyperplans* $\tilde{\mathcal{V}} \subset T\tilde{U}$ *et un* $\tilde{\mathcal{V}}$*-repère de* $T\tilde{U}$*.Alors pour tout* $\tilde{P} \in \Phi^m_{\tilde{\mathcal{V}}}(\tilde{U})$ *l'opérateur* $P = \phi^*\tilde{P}$ *est un* $\Psi_{\mathcal{V}}DO$ *d'ordre* m *sur* U *et on a*

$$c_P(x) = |\phi'(x)| c_{\tilde{P}}(\phi(x)), \qquad x \in U, \tag{34}$$

où $c_P(x)$ *et* $c_{\tilde{P}}(\tilde{x})$ *sont les coefficients des divergences logarithmiques des noyaux de* P *et* $\tilde{P}$ *données par le corollaire 1.15.*

Définition 1.17 *Soit* $P \in \Psi^m_{\mathcal{V}}(U)$ *de symbole principal* $f_m \in S_m(U \times \mathbb{R}^{d+1})$*. On dit que* P *est elliptique dans le* $\Psi_{\mathcal{V}}DO$*-calcul s'il existe* $g_{-m} \in S_{-m}(U \times \mathbb{R}^{d+1})$ *tel que*

$$f_m * g_{-m} = 1 = g_{-m} * f_m. \tag{35}$$

Proposition 1.18 *Soit* $P \in \Psi^m_{\mathcal{V}}(U)$*. Alors:*

1) Le $\Psi_{\mathcal{V}}DO$*-opérateur* P *est elliptique si, et seulement si, il existe* $Q \in \Psi^{-m}_{\mathcal{V}}(U)$ *tel que*

$$PQ = 1 = QP \qquad \mathrm{mod}\ \Psi^{-\infty}(U). \tag{36}$$

2) Si P *est elliptique dans le* $\Psi_{\mathcal{V}}DO$*-calcul c'est un opérateur hypoelliptique, i.e. pour toute* $u \in \mathcal{E}'(U)$ *on a*

$$Pu \text{ lisse près de } x_0 \implies u \text{ lisse près de } x_0. \tag{37}$$

Proposition 1.19 ([BG]) *Soit* Δ *un sous-laplacien sur* U*, i.e.*

$$\Delta = -\sum_{j=1}^{d} X_j^2 - i\lambda(x)X_0 + \sum_{j=1}^{d} \mu_j(x)X_j + \nu(x), \tag{38}$$

où $\lambda, \mu_1, \ldots, \mu_d, \nu$ *sont des fonctions lisses Alors pour chaque* $y \in U$ *il existe un sous-ensemble* $\Lambda^y \subset \mathbb{R}$ *de telle sorte que les assertions suivantes soient équivalentes :*

(i) Pour tout $y \in U$ *le coefficient* $\lambda(y)$ *n'appartient pas à* Λ^y*.*

(ii) Pour tout $y \in U$ *l'opérateur* y*-invariant* Δ_2^y *est inversible.*

(iii) Δ *est elliptique dans le* $\Psi_{\mathcal{V}}DO$*-calcul.*

De plus, si une de ces conditions est satisfaite alors le symbole principal d'une paramétrix est donné par

$$f_{-2}(y,\xi) = f_{-2}^{y}(\xi), \tag{39}$$

où $f_{-2}^{y}(\xi)$ *est de l'inverse de* Δ_2^y.

Soit maintenant $(M, \mathcal{V})$ une variété de Heisenberg et $\mathcal{E}$ un fibré vectoriel au-dessus de M. La proposition 1.18 permet de définir les $\Psi_{\mathcal{V}}DO$ sur M agissant sur les sections de $\mathcal{E}$. Tous les résultats précédents dans le cas d'un ouvert continuent d'être vrais dans le cas d'une variété.

De plus la proposition 1.18 montre aussi que le coefficient $c_P(x)$ d'un $\Psi_{\mathcal{V}}DO$ d'ordre entier P peut-être globalement défini comme une densité sur m à valeurs dans $\operatorname{END}\mathcal{E}$.

Enfin dans le cas d'un $\Psi_{\mathcal{V}}DO$ elliptique sur une variété de Heisenberg compacte on peut construire une paramétrix modulo des opérateurs de rangs finis, de sorte qu'un tel opérateur a un noyau de dimension finie formé de sections lisses et est un opérateur Fredholm.

Proposition 1.20 ([BG]) *Soit* (M^{2n+1}, θ) *une variété pseudo-hermitienne compacte et soit* $\mathcal{V} = \ker\theta$. *Alors les opérateurs suivants sont elliptiques dans le* $\Psi_{\mathcal{V}}DO$*-calcul:*

(i) le laplacien de Kohn $\Box_b$ *agissant sur les* (p,q)*-formes avec* $0 < q < n$*;*

(ii) le sous-laplacien pseudo-hermitien Δ_b*;*

(iii) le sous-laplacien conforme $\boxdot_\theta$.

Noyau de la chaleur d'un sous-laplacien elliptique

Proposition 1.21 ([BGS]) *Soit* $(M^{d+1}, \mathcal{V})$ *une variété de Heisenberg munie d'une densité* > 0 *et soit* Δ *un sous-laplacien positif sur* M.

1) L'opérateur $e^{-t\Delta}$ *est régularisant pour* $t > 0$.

2) Soit $k_t(x,y)$, $t > 0$, *le noyau de* $e^{-t\Delta}$. *Pour* $t \to 0^+$ *on a*

$$k_t(x,x) \sim t^{-\frac{d+2}{2}} \sum_{j\geq 0} a_j(x) t^j, \tag{40}$$

où les $a_j(x)$ *sont des densités lisses sur* M *avec* $a_0(x) > 0$.

3) Pour $t \to 0^+$ *on a*

$$\operatorname{Trace}(e^{-t\Delta}) \sim t^{-\frac{d+2}{2}} \sum_{j\geq 0} t^j \int_M a_j(x). \tag{41}$$

4) Soit $\lambda_k(\Delta)$ la k-ème valeur propre de Δ comptée avec multiplicité. Alors pour k grand on a

$$\lambda_k(\Delta) \sim (Ak)^{\frac{d+2}{2}}, \qquad A = \Gamma(1+\frac{d+2}{2})^{-1} \int_M a_0(x). \tag{42}$$

Dans le cas d'une variété pseudo-hermitienne on obtient :

Proposition 1.22 ([BGS]) *Soit (M^{2n+1}, θ) une variété pseudo-hermitienne compacte. Alors la proposition 1.21 est satisfaite par les opérateurs suivants :*

(i) le laplacien de Kohn $\Box_b$ sur les (p,q)-formes, $0 < q < n$,

(ii) le sous-laplacien pseudo-hermitien Δ_b,

(iii) le sous-laplacien conforme $\boxdot_\theta$.

De plus pour chacun de ces opérateurs le coefficient $a_j(x)$ dans le développement asymptotique (40) est de la forme

$$a_j(x) = A_j(x)(d\theta)^n \wedge \theta, \qquad j \leq 0, \tag{43}$$

où $A_j(x)$ est un polynôme universel en les jets des composantes de la courbure et de la torsion de la connexion de Tanaka-Webster. Pour $j = 0$ et $j = 1$ on a

$$A_0 = \alpha_n, \qquad A_1 = \beta_n R_n, \tag{44}$$

où α_n, β_n sont des constantes universelles et R_n est la courbure scalaire de la connexion de Tanaka-Webster.

Chapitre 2 : $\Psi_{\mathcal{V}}DO$ à paramètre et résolvante d'un sous-laplacien elliptique

Dans ce chapitre on développe un calcul idoine de $\Psi_{\mathcal{V}}DO$ à paramètre qui permet de construire une résolvante asymptotique pour un sous-laplacien elliptique sur une variété de Heisenberg compacte.

Dans tout ce chapitre $\Lambda \subset \mathbb{C}\backslash 0$ est un pseudo-cône ouvert (*cf.* définition ci-dessous).

$\Psi_{\mathcal{V}}DO$-calcul à paramètre

Définition 2.1 *On dit que $\Lambda \subset \mathbb{C}\backslash 0$ est un pseudo-cône si pour tout $t \in (0,1)$ on a $t\Lambda \subset \Lambda$ et s'il existe Θ conique et D borné tels que $\Lambda = \Theta \cup D$.*

Si Λ et Λ' sont deux pseudo-cones on écrit $\Lambda' \subset\subset \Lambda$ pour signifier qu'à l'origine près la fermeture de Λ' est contenue dans l'intérieur de Λ.

L'espace des paramètres est l'espace de Fréchet suivant :

Définition 2.2 $\mathrm{Hol}^p(\Lambda)$, *$p \in \mathbb{Z}$, est l'espace des fonctions holomorphes $h : \Lambda \to \mathbb{C}$ telles que pour tout pseudo-cône $\Lambda' \subset\subset \Lambda$ on ait*

$$|h(\lambda)| \leq C_{\Lambda'}(1+|\lambda|)^p, \qquad \lambda \in \Lambda'. \tag{45}$$

Sa topologie est définie au moyen des semi-normes données par des plus petites constantes dans ces estimées.

Définition 2.3 *Si E espace vectoriel topologique localement convexe $\mathrm{Hol}^p(\Lambda, E)$, $p \in \mathbb{Z}$, est l'espace des $\mathrm{Hol}^p(\Lambda)$-familles à valeurs dans E, c.a.d. des applications holomorphes $h : \Lambda \to E$ telles que, pour toute semi-norme continue p sur $\mathcal{E}$ et tout pseudo-cône $\Lambda' \subset\subset \Lambda$, on ait*

$$|p(h(\lambda))| \leq C_{p\Lambda'}(1+|\lambda|)^p, \qquad \lambda \in \Lambda'. \tag{46}$$

Si $E = S^k_{||}(U \times \mathbb{R}^{d+1})$ (resp. $E = S^{-\infty}(U \times \mathbb{R}^{d+1})$) on utilise la notation $S^{p,k}_{||}(U \times \mathbb{R}^{d+1}, \Lambda)$ (resp. $S^{p,-\infty}(U \times \mathbb{R}^{d+1}, \Lambda)$).

Définition 2.4 *$S^p_m(U \times \mathbb{R}^{d+1}, \Lambda)$, $m, p \in \mathbb{Z}$, consiste en les $\mathrm{Hol}^p(\Lambda)$-familles $f_{(\lambda)}$ de fonctions lisses sur $U \times \mathbb{R}^{d+1}$ telles que*

$$f_{(t^2\lambda)}(x, t.\xi) - t^m f_{(\lambda)}(x, \xi) \in S^{p,-\infty}(U \times \mathbb{R}^{d+1}, \Lambda), \qquad 0 < t < 1. \tag{47}$$

Lemme 2.5 *Soit $m, p \in \mathbb{Z}$ et soit $p_- = \max(0, -p)$. Alors*

$$S^p_m(U \times \mathbb{R}^{d+1}, \Lambda) \subset S^{p,m+p_-}_{||}(U \times \mathbb{R}^{d+1}, \Lambda). \tag{48}$$

Définition 2.6 *$S^{p,m}(U \times \mathbb{R}^{d+1}, \Lambda)$, $m, p \in \mathbb{Z}$, est l'espace des $\mathrm{Hol}^p(\Lambda)$-familles $f_{(\lambda)}$ de fonctions lisses sur $U \times \mathbb{R}^{d+1}$ avec un développement asymptotique*

$$f_{(\lambda)} \sim \sum_{j \geq 0} f_{(\lambda),m-j}, \qquad f_{(\lambda),k} \in S^p_k(U \times \mathbb{R}^{d+1}, \Lambda), \tag{49}$$

au sens où, pour tout entier N, si J est assez grand on a

$$f - \sum_{j \leq J} f_{(\lambda),m-j} \in S^{p,-N}_{||}(U \times \mathbb{R}^{d+1}, \Lambda). \tag{50}$$

Définition 2.7 *1) $\Psi^{p,-\infty}_{\mathcal{V}}(U, \Lambda)$, $p \in \mathbb{Z}$, est l'espace des familles d'opérateurs de $C^\infty_c(U)$ vers $C^\infty(U)$ données par des $Hol^p(\Lambda)$-familles de noyaux lisses.*

2) $\Psi_{\mathcal{V}}^{p,m}(U,\Lambda)$, $m,p\in\mathbb{Z}$, est l'espace des familles $P_{(\lambda)}$ à valeurs dans $\mathcal{L}(C_c^\infty(U),C^\infty(U))$ de la forme

$$P_{(\lambda)} = f_{(\lambda)}(x,\sigma(x,D)) + R_{(\lambda)}, \tag{51}$$

avec $f_{(\lambda)}\in S^{p,m}(U\times\mathbb{R}^{d+1},\Lambda)$, appelé le symbole de $P_{(\lambda)}$, et $R_{(\lambda)}\in\Psi_{\mathcal{V}}^{p,-\infty}(U,\Lambda)$.

Proposition 2.8 *Soit $m,p\in\mathbb{Z}$. On pose $k=m+p_-$ si $m+p_-\geq 0$ ou bien $k=\frac{1}{2}(m+p_-)$ sinon.*

a) La classe $\Psi_{\mathcal{V}}^{p,m}(U,\Lambda)$ ne dépend pas du choix du $\mathcal{V}$-repère $X_0,X_1,\ldots,X_d$.

b) Chaque $P_{(\lambda)}\in\Psi_{\mathcal{V}}^{p,m}(U,\Lambda)$ s'étend en une $\mathrm{Hol}^p(\Lambda)$-famille d'opérateurs continus de $\mathcal{E}'(U)$ vers $\mathcal{D}'(U)$.

c) Le noyau de tout $P_{(\lambda)}\in\Psi_{\mathcal{V}}^{p,m}(U,\Lambda)$ est en dehors de la diagonale par une $\mathrm{Hol}^p(\Lambda)$-famille de fonctions lisses.

d) Pour tout $s\in\mathbb{R}$, chaque $P_{(\lambda)}\in\Psi_{\mathcal{V}}^{p,m}(U,\Lambda)$ définit une $\mathrm{Hol}^p(\Lambda)$-famille d'opérateurs continus de $H^s_{\mathrm{comp}}(U)$ vers $H^{s-k}_{\mathrm{loc}}(U)$.

Un opérateur à paramètre $P_{(\lambda)}:\mathcal{E}'(U)\to\mathcal{D}'(U)$ est dit *uniformèment proprement supporté* s'il est proprement supporté uniformèment par rapport à λ.

Proposition 2.9 *Soit $P_{(\lambda)}\in\Psi_{\mathcal{V}}^{p,m}(U,\Lambda)$. Alors:*

1) On peut écrire $P_{(\lambda)}$ sous la forme $P_{(\lambda)}=Q_{(\lambda)}+R_{(\lambda)}$ avec $Q_{(\lambda)}\in\Psi_{\mathcal{V}}^{p,m}(U,\Lambda)$, uniformèment proprement supporté, et $R_{(\lambda)}$ dans $\Psi_{\mathcal{V}}^{p,-\infty}(U,\Lambda)$.

2) Si $P_{(\lambda)}$ est uniformèment proprement supporté, il définit des $\mathrm{Hol}^p(\Lambda)$-famille d'endomorphismes continus de $C_c^\infty(U)$, $C^\infty(U)$, $\mathcal{E}'(U)$ et $\mathcal{D}'(U)$ respectivement.

Par continuité la convolution $*$ pour les symboles induit une application bilinéaire

$$S_{||}^{p_1,k_1}(U\times\mathbb{R}^{d+1},\Lambda)\times S_{||}^{p_2,k_2}(U\times\mathbb{R}^{d+1},\Lambda)\longrightarrow S_{||}^{p_1+p_2,k_1+k_2}(U\times\mathbb{R}^{d+1},\Lambda), \tag{52}$$

qui ensuite définit une convolution sur $S_*^p(U\times\mathbb{R}^{d+1},\Lambda)$,

$$*: S_{m_1}^{p_1}(U\times\mathbb{R}^{d+1})\times S_{m_2}^{p_2}(U\times\mathbb{R}^{d+1})\longrightarrow S_{m_1+m_2}^{p_1+p_2}(U\times\mathbb{R}^{d+1}). \tag{53}$$

Proposition 2.10 *Soit* $P_{i(\lambda)} \in \Psi_{\mathcal{V}}^{p_i,m_i}(U,\Lambda)$*,* $i=1,2$*, de symbole* $f_{i(\lambda)} \sim \sum f_{i(\lambda),m_1-j}$ *tels que* $P_{1(\lambda)}$ *ou* $P_{2(\lambda)}$ *soit uniformèment proprement supporté. Alors* $P_{1(\lambda)}P_{2(\lambda)}$ *est dans* $\Psi_{\mathcal{V}}^{p_1+p_2,m_1+m_2}(U,\Lambda)$ *et a pour symbole* $f_{(\lambda)} \sim \sum f_{(\lambda),m_1+m_2-j}$*, avec*

$$f_{(\lambda),m_1+m_2-j} = \sum h_{\alpha\beta\gamma\delta}(x) f^{\delta}_{1(\lambda),m_1-k} * f^{\beta\gamma}_{2(\lambda),m_2-l,\alpha}(x,\xi), \tag{54}$$

où la somme est prise sur les indices tels que $|\beta|+|\alpha| \leq \langle\delta\rangle+\langle\beta\rangle-\langle\gamma\rangle = j-k-l$ *et* $|\gamma|=|\beta|$.

Proposition 2.11 *Soit* $\phi : U \to \tilde{U}$ *un difféomorphisme Heisenberg, où* $\tilde{U}$ *est un autre ouvert de* $\mathbb{R}^{d+1}$ *muni d'un sous-fibré en hyperplans* $\tilde{\mathcal{V}} \subset T\tilde{U}$ *et d'un* $\tilde{\mathcal{V}}$*-repère. Alors pour tout* $\tilde{P}_{(\lambda)}$ *dans* $\Psi_{\tilde{\mathcal{V}}}^{p,m}(\tilde{U},\Lambda)$ *la famille d'opérateurs* $P_{(\lambda)} = \phi^*\tilde{P}_{(\lambda)}$ *appartient à* $\Psi_{\mathcal{V}}^{p,m}(U,\Lambda)$.

On peut ainsi définir les $\Psi_{\mathcal{V}}DO$ à paramètre sur n'importe quelle variété de Heisenberg et agissant sur les sections d'un fibré vectoriel. Toutes les propriétés précédemment décrites restent vraies *mutatis standis* pour les variétés.

Résolvante asymptotique pour un sous-laplacien elliptique

Pour tout $R \geq 0$ on pose :

$$\Lambda_R = \{\lambda \in \mathbb{C}\backslash 0;\ \Re\lambda < 0 \text{ or } |\lambda| < R\}. \tag{55}$$

Proposition 2.12 *Soit* Δ *un sous-laplacien elliptique sur* U *de la forme*

$$\Delta = -\sum_{j=1}^{d} X_j^2 - i\nu(x)X_0 + \sum_{j=1}^{d} \mu_j(x)X_j + \eta(x), \tag{56}$$

où ν*,* $\mu_1,\ldots,$ μ_d*,* η *sont des fonctions lisses. Soit* $p_2(x,\xi) = \sum_{j=1}^d \xi_j^2 + i\nu(x)\xi_0$ *le symbole principal de* Δ*. Alors pour tout* $R>0$ *il existe* $f_{(\lambda)} \in S^{-1}_{-2}(U\times\mathbb{R}^{d+1},\Lambda_R)$ *tel que*

$$(p_2-\lambda) * f_{(\lambda)} = 1 = f_{(\lambda)} * (p_2-\lambda) \qquad \bmod S^{-\infty,-\infty}(U\times\mathbb{R}^{d+1},\Lambda_R). \tag{57}$$

Proposition 2.13 *Soit* $(M,\mathcal{V})$ *une variété de Heisenberg et* Δ *un sous-laplacien elliptique sur* M*. Alors, pour tout* $R>0$*, il existe* $Q_{(\lambda)} \in \Psi_{\mathcal{V}}^{-1,-2}(M,\Lambda_R)$ *tel que*

$$(\Delta-\lambda)Q_{(\lambda)} = 1 = Q_{(\lambda)}(\Delta-\lambda) \qquad \bmod \Psi_{\mathcal{V}}^{-1,-\infty}(M,\Lambda_R). \tag{58}$$

Définition 2.14 *Un rayon* $L \subset \mathbb{C}$ *est un rayon de croissance minimale pour* Δ *si* $\Delta-\lambda$ *est inversible pour tout* $\lambda \in L$ *et la norme de la résolvante* $\|(\Delta-\lambda)^{-1}\|$ *est un* $\mathrm{O}(1/|\lambda|)$ *sur* L.

Pour $r > 0$ et Θ secteur angulaire ouvert on pose

$$\Theta_r = \{\lambda \in \Theta; |\lambda| > r\}. \tag{59}$$

Proposition 2.15 *Soit Θ un secteur angulaire ouvert $\subset\subset \Lambda_0$. Alors il existe $r > 0$ tel que le spectre de Δ soit contenu dans $\mathbb{C} \setminus \Theta_r$ et qu'on ait*

$$\|(\Delta - \lambda)^{-1}\| \leq C_{\Theta r}|\lambda|^{-1}, \qquad \lambda \in \Theta_r. \tag{60}$$

Corollaire 2.16 *Tout sous-laplacien elliptique auto-adjoint sur une variété de Heisenberg compacte est borné inférieurement et vérifie donc les conclusions de la proposition 1.21.*

Théorème 2.17 *Soit $(M, \mathcal{V})$ une variété de Heisenberg compacte et soit Δ un sous-laplacien elliptique sur M. Alors il existe $R > 0$ et un pseudo-cône ouvert Λ contenant $D(0, R) \setminus 0$ et contenu dans*

$$\Lambda_R = \{\lambda \in \mathbb{C}\backslash 0;\ \Re\lambda < 0 \ \mathit{or}\ |\lambda| < R\}, \tag{61}$$

de telle sorte que

(i) Pour tout $\lambda \in \Lambda$ l'opérateur $\Delta - \lambda$ soit inversible sur $L^2(M)$.

(ii) La famille $(\Delta - \lambda)^{-1}$, $\lambda \in \Lambda$, appartienne à $\Psi_{\mathcal{V}}^{-1,-2}(M, \Lambda)$.

(iii) Pour tout pseudo-cône $\Lambda' \subset\subset \Lambda$ on ait

$$\|(\Delta - \lambda)^{-1}\| \leq C_{\Lambda'}(1 + |\lambda|)^{-1}, \qquad \lambda \in \Lambda'. \tag{62}$$

En particulier chaque rayon contenu dans Λ est un rayon de croissance minimale pour Δ.

Chapitre 3 : famille holomorphes de $\Psi_{\mathcal{V}}DO$ et puissances complexes d'un sous-laplacien elliptique

Le but de ce chapitre est de définir les familles holomorphes de $\Psi_{\mathcal{V}}DO$ et de construire les puissances complexes d'un sous-laplacien elliptique sur une variété de Heisenberg compacte.

Familles holomorphes de $\Psi_{\mathcal{V}}DO$

Ici Ω est un domaine ouvert de $\mathbb{C}$.

Définition 3.1 *Une famille $(f_z) \subset S^*(U \times \mathbb{R}^{d+1})$ indexée par Ω est holomorphe si les conditions sont satisfaites :*

(i) l'ordre m_z *du symbole* f_z *dépend holomorphiquement de* z *;*

(ii) pour $(x, \xi) \in U \times \mathbb{R}^{d+1}$ *fixé, la fonction* $z \to f_z(x, \xi)$ *est holomorphe sur* Ω *;*

(iii) les bornes du développement asymptotique

$$f_z(x, \xi) \sim \sum_{j \geq 0} f_{z, m_z - j}(x, \xi), \qquad f_{z,l} \in S_l(U \times \mathbb{R}^{d+1}), \tag{63}$$

sont localement uniformes par rapport à z.

L'espace des familles holomorphes de symboles est noté $\mathrm{Hol}(\Omega, S^*(U \times \mathbb{R}^{d+1})$.

Définition 3.2 *Une famille* $(P_z) \subset \Psi^*_{\mathcal{V}}(U)$ *est holomorphe quand* P_z *est de la forme*

$$P_z = f_z(x, \sigma(x, D)) + R_z, \tag{64}$$

avec (f_z) *famille holomorphe de symboles et* (R_z) *famille holomorphe d'opérateurs régularisants. L'espace des familles holomorphes de* $\Psi_{\mathcal{V}}DO$ *est noté* $\mathrm{Hol}(\Omega, \Psi^*_{\mathcal{V}}(U))$.

Proposition 3.3 *Soit* (P_z) *une famille holomorphe de* $\Psi_{\mathcal{V}}DO$ *indexée par* Ω*. Alors :*

1) La famille (P_z) *donne des familles holomorphes à valeurs dans* $\mathcal{L}(C^\infty_c(U), C^\infty(U))$ *et* $\mathcal{L}(\mathcal{E}'(U)), \mathcal{D}'(U))$.

2) Le noyau de P_z *est donné en dehors de la diagonale par une famille holomorphe fonctions lisses.*

3) On peut écrire P_z *sous la forme* $P_z = Q_z + R_z$ *avec* (Q_z) *famille holomorphe de* $\Psi_{\mathcal{V}}DO$ *uniformèment proprement supportés et* (R_z) *famille holomorphe d'opérateurs régularisants.*

4) Si la famille (P_z) *de* $\Psi_{\mathcal{V}}DO$ *est uniformèment proprement supportée, elle induit des familles holomorphes d'endomorphismes continus de* $C^\infty_c(U)$*,* $C^\infty(U)$*,* $\mathcal{E}'(U)$ *and* $\mathcal{D}'(U)$ *respectivement.*

Proposition 3.4 *Soit* $(P_{1,z})$ *et* $(P_{2,z})$ *deux familles holomorphes de* $\Psi_{\mathcal{V}}DO$*, l'une d'entre elle étant uniformèment proprement supportée. Alors la famille* $P_z = P_{1,z}P_{2,z}$ *est une famille holomorphe de* $\Psi_{\mathcal{V}}DO$.

Proposition 3.5 *Soit* $\phi : U \to \tilde{U}$ *un difféomorphisme Heisenberg, où* $\tilde{U}$ *est un autre ouvert de* $\mathbb{R}^{d+1}$ *muni d'un sous-fibré en hyperplans* $\tilde{\mathcal{V}} \subset T\tilde{U}$ *et d'un* $\tilde{\mathcal{V}}$*-repère Alors pour famille* $\tilde{P}_z$ *dans* $\mathrm{Hol}(\Omega, \Psi^*_{\tilde{\mathcal{V}}}(\tilde{U}))$ *la famille* $P_z = \phi^* \tilde{P}_z$ *appartient à* $\mathrm{Hol}(\Omega, \Psi^*_{\tilde{\mathcal{V}}}(\tilde{U}))$.

On peut ainsi définir les familles holomorphes de $\Psi_{\mathcal{V}}DO$ sur n'importe quelle variété de Heisenberg et agissant sur les sections d'un fibré vectoriel. Toutes les propriétés précédentes sont encore vraies pour les variétés.

Puissances complexes d'un sous-laplacien elliptique

Soit Δ un sous-laplacien elliptique sur une variété de Heisenberg $(M^{d+1}, \mathcal{V})$. Supposons d'abord qu'on ait $\Delta \geq c > 0$ par rapport à un produit scalaire définit par une densité non-négative sur M. On peut alors définir les puissances complexes de Δ par calcul fonctionnel comme un groupe à 1 -paramètre d'opérateurs non-bornés sur $L^2(M)$. En utilisant la formule de Mellin et la construction pseudo-différentiel du noyau de la chaleur donnée dans [BGS] on montre :

Théorème 3.6 *La famille $(\Delta^s)_{s\in\mathbb{C}}$ des puissances complexes de Δ est une famille holomorphe de $\Psi_\mathcal{V}DO$ sur M.*

Ne supposons plus Δ non-négatif. Il vérifie de toutes façons les conclusions du théorème 2.17. Soit $R > 0$ et Λ un pseudo-cône tels que dans le théorème 2.17. En particulier chaque rayon contenu dans Λ est un rayon de croissance minimale. Pour simplifier supposons que l'axe réel négatif soit un tel rayon et considérons une courbe $\Gamma \subset \Lambda$ commenant à l'infini, passant le long du rayon $\lambda < 0$ jusqu'à un petit disque centré en l'origine et de rayon $\rho < R$, tournant alors autour de cercle dans le sens indirect puis retournant à l'infini le long du rayon $\lambda < 0$. Pour $\Re s < 0$ on pose :

(65) $$\Delta_s = \frac{1}{2i\pi}\int_\Gamma \lambda^s(\Delta-\lambda)^{-1}d\lambda.$$

Lemme 3.7 *La famille (Δ_s) ci-dessus est une famille holomorphe de $\Psi_\mathcal{V}DO$ t.q.* $\operatorname{ord}\Delta_s = 2s$.

Lemme 3.8 ([Se]) *Supposons Δ inversible. Alors la famille (Δ_s) a les propriétés suivantes :*

1) Elle contient les puissances entières négatives de Δ, c.a.d.

(66) $$\Delta_{-k} = \Delta^{-k} \qquad k \text{ entier} > 0.$$

2) C'est un semi-groupe, i.e.

(67) $$\Delta_s\Delta_t = \Delta_{s+t} \qquad \Re s < 0, \quad \Re t < 0.$$

Définition 3.9 *Supposons Δ inversible. Alors Δ^s, $s \in \mathbb{C}$, est définit par*

(68) $$\Delta^s = \Delta^k\Delta_{s-k},$$

où k est entier $> \Re s$ dont la valeur est indifférente.

On obtient ainsi :

Théorème 3.10 *Supposons Δ inversible. Alors la famille (Δ^s) des puissances complexes de Δ est un groupe à 1-paramètre de $\Psi_{\mathcal{V}}DO$ tel que $\Delta^0 = 1$ et $\Delta^1 = \Delta$.*

Supposons maintenant Δ auto-adjoint mais non inversible. Alors la propriété de semi-groupe (67) reste vraie. L'égalité (66) aussi à condition qu'on remplace les inverses par les inverses partiels. On peut alors encore définir les puissances complexes de Δ et dans ce cas on obtient :

Théorème 3.11 *Supposons Δ auto-adjoint. Alors la famille (Δ^s) des puissances complexes de Δ est un groupe à 1-paramètre de $\Psi_{\mathcal{V}}DO$ tel que $\Delta^1 = \Delta$ et $\Delta^0 = 1 - \Pi_0$, où Π_0 est le projecteur orthogonal sur* $\ker \Delta$.

Chapitre 4 : Résidu non commutatif pour les variétés de Heisenberg

Dans ce chapitre on construit un prolongement analytique de la trace sur les $\Psi_{\mathcal{V}}DO$ d'ordre complexe non entier, comme dans [KV] et [CM2], et on montre qu'on obtient alors sur les $\Psi_{\mathcal{V}}DO$ d'ordre non entier une trace résiduelle qui est un résidu non-commutatif pour le $\Psi_{\mathcal{V}}DO$-calcul. On montre ensuite que ce nouveau résidu non-commutatif permet d'étendre la trace de Dixmier à toute l'algèbre des $\Psi_{\mathcal{V}}DO$ d'ordres entiers et que, si la variété est connexe, il induit l'unique trace à coefficient multiplicatif près sur cette algèbre quotientée par les opérateurs régularisants.

Régularisation de la trace et résidu non-commutatif

Soit $(M^{d+1}, \mathcal{V})$ une variété de Heisenberg compacte et $\mathcal{E}$ un fibré vectoriel au-dessus de M. On va montrer que la fonctionnelle Trace qui est *a priori* définie pour les $\Psi_{\mathcal{V}}DO$ qui sont dans

$$\Psi_{\mathcal{V}}^{\text{int}}(M, \mathcal{E}) = \{P \in \Psi_{\mathcal{V}}^*(M, \mathcal{E}); \Re \text{ord} P < -(d+2)\}, \tag{69}$$

a un prolongement analytique sur

$$\Psi_{\mathcal{V}}^{\mathbb{C}\backslash\mathbb{Z}} = \{P \in \Psi_{\mathcal{V}}^*(M, \mathcal{E}); \text{ord} P \in \mathbb{C}\backslash\mathbb{Z}\}. \tag{70}$$

Le point de départ est de réinterpréter le lemme 1.10 en termes de familles holomorphes.

Lemme 4.1 *Pour $f \in S_{\mathbb{C}\backslash\mathbb{Z}}(\mathbb{R}^{d+1})$ on note τ_f son unique extension homogène comme distribution sur $\mathbb{R}^{d+1}$ donnée par le lemme 1.10. Alors :*

1) L'application $f \to \tau_f$ est holomorphe de $S_{\mathbb{C}\backslash\mathbb{Z}}(\mathbb{R}^{d+1})$ vers $\mathcal{S}'(\mathbb{R}^{d+1})$.

2) Soit (f_z) une famille holomorphe de symboles telle que $\operatorname{ord} f_z = z$*. Alors pout toute $u \in \mathcal{S}(\mathbb{R}^{d+1})$ la fonction $z \to \langle \tau_{f_z}, u \rangle$ a au plus des singularités de type pôle simple près de $\mathbb{Z}$ dont les résidus sont donnés par*

$$\operatorname{res}_{z=k} \langle \tau_{f_z}, u \rangle = \sum_{<\alpha>=-(k+d+2)} \frac{1}{\alpha!} c_\alpha(f_k) u^{(\alpha)}(0), \qquad k \in \mathbb{Z}, \tag{71}$$

où les $c_\alpha(f_k)$ sont les obstructions à une extension homogène de f_k données par le lemme 1.10.

Soit L la fonctionnelle sur $S^{\text{int}}(\mathbb{R}^{d+1})$ definie par

$$L(f) = \int f(\xi) d\xi, \qquad f \in S^{\text{int}}(\mathbb{R}^{d+1}). \tag{72}$$

Lemme 4.2 *1) La fonctionnelle L ci-dessus a un unique prolongement holomorphe $\tilde{L}$ sur $S^{\mathbb{C}\setminus\mathbb{Z}}(\mathbb{R}^{d+1})$. La valeur de $\tilde{L}$ en le symbole $f \sim \sum f_{m-j}$ d'ordre non entier est donnée par*

$$\tilde{L}(f) = \int (f(\xi) - \sum_{j \leq N} \tau_{m-j}(\xi)) d\xi, \qquad N \geq \Re m + d + 2, \tag{73}$$

où τ_{m-j} est l'unique extension homogène de f_{m-j}.

2) Soit (f_z) une famille holomorphe à valeurs dans $S^*(\mathbb{R}^{d+1})$ telle que $\operatorname{ord} f_z = z$*. Alors $\tilde{L}(f_z)$ a au plus des pôle simples près de $\mathbb{Z}$ de résidus*

$$\operatorname{res}_{z=k} \tilde{L}(f_z) = -c_0(f_{k,-(d+2)}) = -\int_{\|\xi\|=1} f_{k,-(d+2)}(\xi) i_E d\xi, \qquad k \in \mathbb{Z}. \tag{74}$$

De la démonstration de lemme on en obtient directement la version C^∞ suivante.

Lemme 4.3 *Soit U un ouvert de $\mathbb{R}^{d+1}$.*

1) L'application $f \to \tilde{L}(f(x,.))$ est holomorphe de $S^{\mathbb{C}\setminus\mathbb{Z}}(U \times \mathbb{R}^{d+1})$ vers $C^\infty(U)$.

2) Soit (f_z) une famille holomorphe à valeurs dans $S^*(U \times \mathbb{R}^{d+1})$ telle que $\operatorname{ord} f_z = z$*. Alors $\tilde{L}(f_z(x,.))$ est méromorphe pour la topologie C^∞.*

Théorème 4.4 *1) La fonctionnelle* Trace *on $\Psi_{\mathcal{V}}^{\text{int}}(M, \mathcal{E})$ a un unique prolongement holomorphe sur $\Psi_{\mathcal{V}}^{\mathbb{C}\setminus\mathbb{Z}}(M, \mathcal{E})$ défini par*

$$\operatorname{TR} P = \int_M \operatorname{tr}_{\mathcal{E}} t_P(x), \qquad P \in \Psi_{\mathcal{V}}^{\mathbb{C}\setminus\mathbb{Z}}(M, \mathcal{E}), \tag{75}$$

où $t_P(x)$ est une densité sur M à valeurs dans $\operatorname{END} \mathcal{E}$ *invariante par difféomorphismes Heisenberg.*

2) Soit P_1 et P_2 dans $\Psi_{\mathcal{V}}^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})$ tels que $\operatorname{ord} P_1 + \operatorname{ord} P_2 \notin \mathbb{Z}$. *Alors*

$$\mathrm{TR}\, P_1 P_2 = \mathrm{TR}\, P_2 P_1. \tag{76}$$

3) Soit $P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E})$ et soit (P_z) une famille holomorphe de $\Psi_{\mathcal{V}}DO$ telle que $P_0 = P$ et $\operatorname{ord} P_z = z + \operatorname{ord} P$. *Alors* $\mathrm{TR}\, P_z$ *a au plus un pôle simple en $z = 0$ et on a*

$$\operatorname{res}_{z=0} \mathrm{TR}\, P_z = -\int_M \operatorname{tr}_{\mathcal{E}} c_P(x), \tag{77}$$

où $c_P(x)$ est la densité sur M qui apparaît comme le coefficient de la singularité logarithmique du noyau de P près de la diagonale.

On définit alors le résidu non-commutatif pour les variétés de Heisenberg comme suit.

Définition 4.5 *Le résidu non-commutatif sur $\Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E})$ est la fonctionnelle linéaire*

$$\operatorname{Res} P = \int_M c_P(x), \qquad P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E}). \tag{78}$$

Proposition 4.6 *1) Soit $P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E})$ et (P_z) une famille holomorphe à valeurs dans $\Psi_{\mathcal{V}}^{*}(M,\mathcal{E})$ alors $P_0 = P$ et* $\operatorname{ord} P_z = z + \operatorname{ord} P$. *Alors*

$$\operatorname{Res} P = -\operatorname{res}_{z=0} \mathrm{TR}\, P_z. \tag{79}$$

En particulier, si Δ est sous-laplacien elliptique sur M on a

$$\operatorname{Res} P = \operatorname{res}_{z=0} \mathrm{TR}\, P\Delta^{-z/2}, \qquad P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E}). \tag{80}$$

2) La fonctionnelle Res *est une trace sur $\Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E})$ s'annulant sur les $\Psi_{\mathcal{V}}DO$ d'ordre entier $< -(d+2)$.*

3) Soit $\phi : (M,\mathcal{V}) \rightarrow (\tilde{M},\tilde{\mathcal{V}})$ un difféomorphisme Heisenberg. Alors

$$\operatorname{Res} \phi_* P = \operatorname{Res} P, \qquad P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E}). \tag{81}$$

Trace de Dixmier des $\Psi_{\mathcal{V}}DO$

Soit $(M^{d+1},\mathcal{V})$ une variété de Heisenberg compacte et $\mathcal{E}$ un fibré vectoriel au-dessus de M.

Théorème 4.7 *1) Soit $P \in \Psi_{\mathcal{V}}^{m}(M,\mathcal{E})$ avec $-k = \Re m < 0$. Alors*

$$\mu_n(P) = \mathrm{O}(n^{-\frac{k}{d+2}}) \qquad \text{quand } n \rightarrow \infty. \tag{82}$$

2) Tout $P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E})$ d'ordre $-(d+2)$ est mesurable pour la trace de Dixmier au sens de [Co1] et on a

$$⨍ P = \frac{1}{d+2} \operatorname{Res} P. \tag{83}$$

Traces et sommes de commutateurs sur l'algèbre des $\Psi_{\mathcal{V}}DO$

On suppose dans cette section que M est connexe. Alors :

Théorème 4.8 *Toute trace sur $\Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E})/\Psi^{-\infty}(M,\mathcal{E})$ est proportionnelle au résidu non-commutatif.*

Corollaire 4.9 *Soit $P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E})$. Alors P est une somme de commutateurs si, et seulement si, il est de la forme $P = Q + R$ avec $Q \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E})$ et $R \in \Psi^{-\infty}(M,\mathcal{E})$ tels que* $\operatorname{Res} Q = \operatorname{Trace} R = 0$.

Chapitre 5 : géométrie spectrale des variétés de Heisenberg et pseudo-hermitienne

Dans ce chapitre on donne des applications géométriques du résidu non commutatif et de la trace régularisée. D'abord on définit la fonction zêta d'un sous-laplacien elliptique dans le $\Psi_{\mathcal{V}}DO$-calcul et on relie ses résidus et valeurs régulières aux coefficients du développement de la chaleur. On obtient ensuite des formules variationelles pour les fonctions zêta qu'on utilise pour produire des invariants conformes d'une variété pseudo-hermitienne. Après on étudie la géométrie non-commutative des variétés de Heisenberg et en particulier on définit l'aire d'une variété pseudo-hermitienne de dimension 3. Enfin on donne des formules locales pour calculer l'indice d'une racine carrée d'un sous-laplacien elliptique. D'abord on montre qu'en dimension paire l'indice est toujours égala à zéro et qu'en dimension impaire il est donné par l'intégrale de la densité qui apparaît comme le terme constant dans l'asymptotique du noyau de la chaleur du sous-laplacien. Ensuite, en utilisant la cohomologie cyclique et la formule d'indice locale de Connes-Moscovici [CM2], on montre qu'il existe un courant de Rham, calculable par des formules locales explicites et dont l'accouplement avec le caractère de Chern donne l'indice à coefficients dans la K-théorie.

Fonction zêta d'un sous-laplacien elliptique

Soit $(M^{d+1},\mathcal{V})$ une variété de Heisenberg compacte et soit Δ un sous-laplacien elliptique sur M. On suppose ici qu'il est soit inversible, soit auto-adjoint. On peut alors construire ses puissances complexes et définir la fonction zêta de Δ en posant :

$$\zeta(s) = \operatorname{TR} \Delta^{-s}, \qquad s \in \mathbb{C}. \tag{84}$$

Proposition 5.1 *Soit $\Sigma = \{\frac{1}{2}k\,;\, k = 1,\ldots,d+2\} \cup (-\frac{1}{2} + \mathbb{Z}_-)$. Alors la fonction $\zeta(s)$ est holomorphe sur $\mathbb{C}\setminus\Sigma$ et a au plus des pôles simples sur Σ dont les résidus sont donnés par*

$$\operatorname{res}_{s=s'} \zeta(s) = 2 \operatorname{Res} \Delta^{-s'} = 2\int_M c_{\Delta^{-s'}}(x), \qquad s' \in \Sigma. \tag{85}$$

On suppose désormais que Δ est auto-adjoint. On peut alors relier les résidus et les valeurs régulières de sa fonction zêta aux coefficients de son développement de la chaleur pour t petit

$$k_t(x,x) \sim t^{-\frac{d+2}{2}} \sum_{j\geq 0} t^j a_j(\Delta)(x), \tag{86}$$

où les $a_j(\Delta)(x)$ sont des densités lisses sur M.

Théorème 5.2 *On suppose* $\dim M = d+1$ *impaire,* $d+1 = 2n+1$*. Alors :*

1) Pour $k = 1, \ldots, n+1$ *on a*

$$\mathrm{res}_{s=k}\, \zeta(s) = \frac{1}{2} \operatorname{Res} \Delta^{-k} = \frac{1}{(k-1)!} \int_M a_{n+1-k}(\Delta)(x). \tag{87}$$

2) En $s = 0$ *la valeur régulière est*

$$\zeta(0) = \int_M a_{n+1}(\Delta)(x) - \dim \ker \Delta. \tag{88}$$

3) Pour tout entier non-négatif $-k$ *on a*

$$\zeta(-k) = (-1)^{k-1}(k-1)! \int_M a_{n+1+k}(\Delta)(x). \tag{89}$$

Remarque 5.3 Les calculs sont purement locaux et en fait on a

$$c_{\Delta^{-k}}(x) = \frac{1}{(k-1)!} a_{n-k}(\Delta)(x), \qquad k = 0, \ldots, n, \tag{90}$$

$$t_{\Delta^k}(x) = (-1)^{k-1}(k-1)! a_{n+1+k}(\Delta)(x), \qquad k \in \mathbb{N}^*. \tag{91}$$

Dans le cas pair on obtient :

Théorème 5.4 *Supposons* $\dim M = d+1$ *paire,* $d+1 = 2n$*. Alors :*

1) Pour $k = -n, -n+1, \ldots$ *on a*

$$\mathrm{res}_{s=\frac{1}{2}-k} = \operatorname{Res} \Delta^{-\frac{1}{2}+k} = \frac{1}{\Gamma(\frac{1}{2}-k)} \int_M a_{n+k}(x). \tag{92}$$

2) La valeur régulière en $s = 0$ *est donnée par*

$$\zeta(0) = -\dim \ker \Delta. \tag{93}$$

3) Les valeurs régulières aux entiers strictement positifs sont toutes nulles.

Formules variationnelles et invariance par homotopie

Dans cette section montre des formules variationelles pour la fonction zêta d'un sous-laplacien elliptique. Pour cela on introduit la notion de famille C^1 de $\Psi_{\mathcal{V}}DO$ indexée par un intervalle ouvert I de $\mathbb{R}$.

Définition 5.5 *Une famille (f_ϵ) à valeurs dans $S^m(\mathbb{R}^{d+1})$, $m \in \mathbb{C}$, est C^1 si*

(i) Pour ξ fixé, $f_\epsilon(\xi)$ est une fonction C^1 de ϵ.

(ii) Les symboles homogènes $f_{\epsilon,m-j}$, $j \geq 0$, de f_ϵ sont C^1 avec ϵ.

(iii) Les bornes du développement asymptotique $f_\epsilon \sim \sum f_{\epsilon,m-j}$ sont uniformes pour la topologie C^1.

On peut aussi définir des familles C^1 de symboles sur $U \times \mathbb{R}^{d+1}$, c.a.d des des $C^1(I)\hat{\otimes}C^\infty(U)$-familles de symboles sur $\mathbb{R}^{d+1}$. Cela permet alors de définir les familles C^1 de $\Psi_{\mathcal{V}}DO$.

Si $\Omega \subset \mathbb{C}$ est un ouvert et $\Lambda \subset \mathbb{C}\backslash 0$ un pseudo-cône on définit similairement les $C^1(I)\hat{\otimes}\operatorname{Hol}(\Omega)$-familles et les $C^1(I)\hat{\otimes}\operatorname{Hol}^p(\Lambda)$-familles de $\Psi_{\mathcal{V}}DO$.

Proposition 5.6 *Soit $(M, \mathcal{V})$ une variété de Heisenberg compacte et soit $(P_{\epsilon,s})$ une $C^1\hat{\otimes}\operatorname{Hol}$-famille de $\Psi_{\mathcal{V}}DO$ sur M.*

1) Si $\operatorname{ord} P_{\epsilon,s} \notin \mathbb{Z}$ alors $\operatorname{TR} P(\epsilon,s)$ est holomorphe pour la topologie C^1 et on a

$$\partial_\epsilon \operatorname{TR} P_{\epsilon,s} = \operatorname{TR} \partial_\epsilon P_{\epsilon,s}. \tag{94}$$

2) Supposons que $\operatorname{ord} P_{\epsilon,s} = z$ près d'un entier m. Alors $\operatorname{TR} P_{\epsilon,s}$ a un pôle simple pour la topologie C^1 près de $z = m$ et (94) donne une égalité de fonctions près de $z = m$.

Proposition 5.7 *Soit $(M^{d+1}, \mathcal{V})$ une variété de Heisenberg compacte et soit (Δ_ϵ) une famille C^1 de sous-laplaciens elliptiques sur M. On fait les hypothèses suivantes :*

(i) Δ_ϵ est soit inversible, soit auto-adjoint;

(ii) il existe un pseudo-cône $\Lambda \subset \mathbb{C} \setminus 0$ ouvert et connexe tel que $\Lambda \cap \operatorname{sp} \Delta_\epsilon = \emptyset$ pour tout ϵ.

On définit alors les puissances complexes de Δ_ϵ au moyen d'un contour contenu dans Λ. Alors la fonction $\operatorname{TR} \Delta_\epsilon^{-s}$ est méromorphe pour la topologie C^1 et on a

$$\partial_\epsilon \operatorname{TR} \Delta_\epsilon^{-s} = -s \operatorname{TR} \partial_\epsilon(\Delta_\epsilon)\Delta_\epsilon^{-s-1}. \tag{95}$$

En particulier,

$$\partial_\epsilon \operatorname{Res} \Delta_\epsilon^{-k} = -k \operatorname{Res} \partial_\epsilon(\Delta_\epsilon)\Delta_\epsilon^{-k-1}, \qquad k = 0, \frac{1}{2}, \ldots, \frac{d+2}{2}. \tag{96}$$

Invariants conforme de sous-laplaciens

Théorème 5.8 *Soit* (M^{2n+1},θ) *une variété pseudo-hermitienne compacte et soit* $f\in C^\infty(M)$. *Alors :*

1) On a $a_n(\boxdot_{e^{2f}\theta})(x)=e^{2f(x)}a_n(\boxdot_\theta)(x)$, *i.e.* $a_n(\boxdot_{e^{2f}\theta})(x)$ *est un invariant conforme local de poids* -2.

2) On a l'égalité suivante de fonctions méromorphes

$$\delta_f \operatorname{TR}\boxdot_\theta^{-s}=2s\operatorname{TR} f\,\boxdot_\theta\,. \tag{97}$$

Par conséquent $\zeta_{\boxdot_\theta}(0)$ *est un invariant conforme.*

3) Pour tout entier k,

$$\delta_f\int_M a_k(\boxdot_\theta)(x)=2(n+1-k)\int_M f(x)a_k(\boxdot_\theta)(x). \tag{98}$$

Ainsi $A_{n+1}=\int_M a_{n+1}(\boxdot_{e^{2f}\theta})(x)$ *est un invariant conforme.*

Remarque 5.9 La dernière assertion répond positivement à une conjecture de Branson-Ørsted [BØ2].

Géométrie non commutative des variété pseudo-hermitiennes

Soit (M^{2n+1},θ) une variété pseudo-hermitienne compacte. Alors la proposition 1.22 et le théorème 5.2 permettent d'exprimer les résidus non-commutatifs des sous-laplaciens géométriques comme des intégrales de polynômes universels en les jets des composantes de la courbure et de la torsion de la connexion de Tanaka-Webster connection. Ainsi :

Proposition 5.10 *Soit* Δ_b *le sous-laplacien pseudo-hermitien sur* (M,θ). *Alors*

$$\operatorname{Res}\Delta_b^{-(n+1)}=\alpha_n\int_M(d\theta)^n\wedge\theta,\qquad \operatorname{Res}\Delta_b^{-n}=\beta_n\int_M R_n(d\theta)^n\wedge\theta, \tag{99}$$

où α_n *et* β_n *sont des constantes universelles et* R_n *est la courbure scalaire de Tanaka-Webster.*

Par la remarque 5.3 on a en fait

$$c_{\Delta_b^{-(n+1)}}(x)=\alpha_n(d\theta)^n\wedge\theta(x),\qquad c_{\Delta_b^{-n}}(x)=\beta_nR_n(d\theta)^n\wedge\theta(x). \tag{100}$$

D'où :

Corollaire 5.11 *Pour tout* $f\in C^\infty(M)$ *on a*

$$⨍ f\Delta_b^{-(n+1)}=\alpha_n\int_M f(d\theta)^n\wedge\theta. \tag{101}$$

Ainsi extrapolant à partir de [Co4] on peut interpréter $ds = \alpha_n^{\frac{-1}{2n+2}} \Delta_b^{\frac{1}{2}}$ comme un élément de longueur et définir l'aire de (M, θ) comme suit.

Définition 5.12 *L'aire de* (M, θ) *est*

$$\text{Area}_\theta M = \text{Res}\, ds^2 = \alpha_n^{\frac{-1}{n+1}} \text{Res}\, \Delta_b^{-1} \tag{102}$$

Théorème 5.13 *Pour toute variété pseudo-hermitienne* (M, θ) *de dimension 3 on a*

$$\text{Area}_\theta M = frac18\sqrt{2} \int_M R_1 d\theta \wedge \theta. \tag{103}$$

Par exemple pour la sphère $S^3 \subset \mathbb{C}^2$ avec la forme de contact $\theta = \frac{i}{2}(z_1 d\bar{z}_1 + z_2 d\bar{z}_2)$ on a aire$_\theta = \frac{\pi^2}{2\sqrt{2}}$.

Formules d'indices locales

Soit D un $\Psi_\mathcal{V}DO$ auto-adjoint d'ordre 1 sur une variété de Heisenberg compacte $(M^{d+1}, \mathcal{V})$ et agissant sur les sections d'un fibré vectoriel $\mathcal{S}$ au-dessus de M. On suppose que D^2 est un sous-laplacien elliptique et que D anti-commute avec une $\mathbb{Z}_2$-graduation γ sur $\mathcal{S}$. Par rapport à cette graduation on peut décomposer $\mathcal{S}$ en une somme directe

$$\mathcal{S} = \mathcal{S}^+ \otimes \mathcal{S}^-, \tag{104}$$

et écrire D sous la forme

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \qquad D_\pm : \mathcal{S}_\pm \to \mathcal{S}_\mp. \tag{105}$$

L'indice de D est par définition

$$\text{ind}\, D = \text{ind}\, D^+ = \dim\ker D^+ - \dim\ker D^-, \tag{106}$$

Comme D^2 est un sous-laplacien elliptique il admet un développement de la chaleur

$$k_t(x, x) \sim t^{-\frac{d+2}{2}} \sum_{j \geq 0} t^j a_j(D^2)(x), \tag{107}$$

où les $a_j(D^2)(x)$ sont des densités lisses M à valeurs dans $\text{END}\, \mathcal{S}$.

Théorème 5.14 *1) Si* $d + 1$ *est pair, on a* $\text{ind}\, D = 0$.

2) Si $d + 1 = 2n + 1$ *est impair, alors*

$$\text{ind}\, D = \int_M \text{Str}_\mathcal{S}\, a_0(D^2)(x). \tag{108}$$

Maintenant soit $\mathcal{E}$ un fibré hermitien au-dessus de M et ∇ une connection hermitienne sur $\mathcal{E}$, i.e.

$$\langle \nabla\xi, \eta\rangle - \langle \xi, \nabla\eta\rangle = d\langle \xi, \eta\rangle, \qquad \xi, \eta \in C^\infty(M, \mathcal{E}). \tag{109}$$

On forme le twist de D by ∇ comme suit. Par [Co1, prop. VI.1.4] on définit un morphisme de $C^\infty(M)$-modules $\pi : C^\infty(M, \Lambda T^*M) \to C^\infty(M, \operatorname{End}\mathcal{S})$ en posant

$$\pi(f^0 df^1 \dots df^n) = f^0[D, f^1]\dots[D, f^n], \qquad f^j \in C^\infty(M). \tag{110}$$

Il en résulte ainsi un morphisme de $C^\infty(M)$-modules $\pi : C^\infty(M, \mathcal{S}\otimes\Lambda T^*M) \to C^\infty(M, \operatorname{End}\mathcal{S})$. L'opérateur $D_{\nabla,\mathcal{E}}$ est alors l'opérateur différentiel agissant sur $C^\infty(M, \mathcal{S}\otimes\mathcal{E})$ donné par

$$D_{\nabla,\mathcal{E}} = D \otimes 1 + \pi\nabla, \tag{111}$$

où $\pi\nabla$ est défini au moyen de la composition

$$C^\infty(M, \mathcal{S}\otimes\mathcal{E}) \xrightarrow{1\otimes\nabla} C^\infty(M, \mathcal{S}\otimes T^*M\otimes\mathcal{E}) \xrightarrow{\pi\otimes 1} C^\infty(M, \mathcal{S}\otimes\mathcal{E}). \tag{112}$$

Théorème 5.15 *1) Il existe une classe d'homologie paire* $\operatorname{Ch}_* D \in H_{ev}(M)$ *telle que pour tout fibré hermitien* $\mathcal{E}$ *au-dessus de* M *avec un connexion* ∇ *on ait*

$$\operatorname{ind} D_{\nabla,\mathcal{E}} = \langle \operatorname{Ch}_* D, \operatorname{Ch}^* \mathcal{E}\rangle. \tag{113}$$

2) On définit explicitement un courant de Rham pair $C = (C_{2n})$ *représentant* $\operatorname{Ch}_* D$ *comme suit. Pour* $n \neq 0$ *on définit* C_{2n} *par*

$$\begin{aligned}&\langle C_{2n}, f^0 df^1 \wedge \ldots \wedge df^{2n}\rangle \\ &= (2n)! \sum_\alpha c_\alpha \operatorname{Res} \gamma f^0 [D, f^1]^{\alpha_1} \dots [D, f^{2n}]^{\alpha_{2n}} |D|^{-2(|\alpha|+n)},\end{aligned} \tag{114}$$

où $c_\alpha^{-1} = (-1)^{|\alpha|} 2\alpha!(\alpha_1 + 1)\cdots(\alpha_1 + \cdots + \alpha_{2n} + 2n)$ *et le symbole* $T^{(k)}$ *dénote le k-ème commutateur itéré avec* D^2*; tandis que for* $n = 0$ *on pose*

$$\langle C_0, f\rangle = \int_M f(x) \operatorname{Str}_{\mathcal{S}} a_0(D^2)(x), \tag{115}$$

Introduction

In this thesis dissertation we prove various geometric results for pseudohermitian manifolds, and more generally for Heisenberg manifolds, all coming around with a non-commutative residue within the Heisenberg calculus of [BG] and [Tay]. These include non-commutative geometry, zeta functions, conformal invariants and local index formulae.

Before presenting the articulation of the dissertation, we shall give a short overview of non-commutative residue for classical pseudodifferential operators (ΨDO's). The non-commutative residue ([Wo1], [Kass], [Wo2]) is a trace on the algebra of ΨDO's on a compact manifold M. If P is a ΨDO on M its non-commuatative residue is defined by

$$\operatorname{Res} P = 2 \operatorname{res}_{z=0} \operatorname{Trace} P\Delta^{-z}, \tag{116}$$

where Δ is a Laplacian on M.

In dimension 1 it was found by Manin [Ma] and Adler [Ad] in the context of complete integrable systems. In arbitrary dimension it was discovered by Wodzicki [Wo1] using extensively Seeley's work [Se], while Guillemin [Gu1] studied independently the restriction of Res on ΨDO's with order $\leq -\dim M$. Indeed Wodzicki gave in [Wo2] a complete account on the subject using the formalism of symplectic cones in the spirit of [Gu1].

In fact it follows from [KV] and [CM2] that we can construct the non-commutative residue using only homogeneous distributions and basic properties of ΨDO's (e.g. [Ho3, section 3.2] and [Ho4, sec. 18.1]).

Let M be a compact manifold of dimension n. If P is a ΨDO with $\Re \operatorname{ord} P < -n$, the restriction of the kernel of P on the diagonal is a smooth density $k_P(x,x)$ on M, so that P is traceable and we have

$$\operatorname{Trace} P = \int_M k_P(x,x). \tag{117}$$

In [KV] and [CM2] it is shown that the map $P \rightarrow k_P(x,x)$ with values in the space of smooth densities on M can be extended to a map $P \rightarrow t_P(x)$ on ΨDO's with non-integral complex order and this map is anlytic in some sense. Then an integration over M provides us with a continuation of the trace,

$$\mathrm{TR}\, P = \int_M t_P(x), \qquad \mathrm{ord} P \notin \mathbb{Z}, \tag{118}$$

which is also analytic.

The density $t_P(x)$ is defined by means of homogeneous extensions as distributions of the symbols of P. Such extensions exist if the degree of homogeneity is not an integer. For integral degrees there are obstructions to such extensions, which in turn imply that the kernel of a ΨDO operator P with integral order k has a logarithmic divergency near the diagonal,

$$k_P(x,y) = \sum_{-(n+k)}^{0} a_j(x, x-y) - c_P(x) \log|x-y| + \mathrm{O}(1), \tag{119}$$

with $a_j(x,z)$ homogeneous of degree j in the second variable and $c_P(x)$ is given by

$$c_P(x) = (2\pi)^{-n} \int_{S^{n-1}} f_{-n}(x,\xi) d^{n-1}\xi, \tag{120}$$

where $f_{-n}(x,\xi)$ is the symbol of degree $-n$ of P. Indeed (119) implies that $c_P(x)$ can be globally defined as a density on M.

Moreover one can show ([KV], [CM2]) that given a ΨDO operator P with order $k \in \mathbb{Z}$ and a holomorphic family (P_z) of ΨDO's near $z=0$ such that $P_0 = P$ and $\mathrm{ord} P_z = z + k$ then the function $\mathrm{TR}\, P_z$ has at most a simple pole singularity at $z=0$ with residue

$$\mathrm{res}_{z=0}\, \mathrm{TR}\, P_z = -\int_M c_P(x). \tag{121}$$

Taking $P_z = P\Delta^{\frac{z}{2}}$ we find

$$\mathrm{Res}\, P = \int_M c_P(x). \tag{122}$$

This last equality shows that Res is local and combining with (121) we deduce that the noncommutative is a trace on the integral ΨDO algebra. In fact one can show ([Wo1], [BrGe], [FGLS]) that if M is connected it is the only trace up to constant multiples on the ΨDO algebra quotiented by smoothing operators.

Moreover the non-commutative residue has important applications on mathematics and mathematical physics and play a central role in non-commutative geometry (see [Co1], [CM2], [Ger], [Kast], [KrKh], [KaW]). One the deepest concerns cyclic cohomology and local index formula ([Co1], [CM2]) and we shall briefly present it now.

Consider a non-commutative space represented by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an involutive algebra represented in the Hilbert space $\mathcal{H}$ and D is a selfadjoint operator on $\mathcal{H}$ almost commuting with $\mathcal{A}$, in the sense that $[D, a]$ is a bounded operator on $\mathcal{H}$ for any $a \in \mathcal{A}$. Then ([At], [Kas]) showed that the *datum* of D defined an index map

$$\text{ind}_D : K_*(\mathcal{A}) \longrightarrow \mathbb{Z}, \tag{123}$$

where $K_*(\mathcal{A})$ is the K-theory of the algebra $\mathcal{A}$. This index map can be computed by the Chern character $\text{Ch}_*(D)$ in cyclic cohomology,

$$\text{ind}_D([\mathcal{E}]) = \langle \text{Ch}_*(D), [\mathcal{E}] \rangle, \tag{124}$$

where $\langle ., . \rangle$ denotes the pairing of cyclic cohomology with K-theory.

If the spectral triple has a simple and discrete dimension spectrum Connes and Moscovici [CM2] derived a local formula for the cyclic cocycle $\text{Ch}_*(D)$ as a finite universal linear combination of terms of the form

$$\text{Res}\, a^0 [D, a^1]^{(k_1)} \cdots [D, a^n]^{(k_n)} |D|^{-n-2|k|}, \qquad a^j \in \mathcal{A}, \tag{125}$$

where for an operator T on $\mathcal{H}$ the symbol $T^{(k)}$ denotes the k'th iterated commutator with D^2 and Res is an algebraic analogue of the non-commutative residue on the ΨDO algebra generated by $\mathcal{A}$ and D. In the case a compact spin Riemannian manifold M and $\partial\!\!\!/$ the Dirac operator acting on L^2-spinors, this reduces to the local Atiyah-Singer index theorem ([AS]).

However the computation for the transversally elliptic signature operator in [CM2, sec. I.2] is a rather formidable task, even for codimension 1 foliations. This led Connes and Moscovici [CM3] to invent in cyclic cohomology for Hopf algebras. Then they showed that the index took place within the cyclic cohomology of universal Hopf algebras which can be related to Gel'fand-Fuchs cohomology. Note that the Hopf algebras involved are very similar to those introduced by D. Kreimer and A. Connes in the context of quantum fields theories ([CK1], [CK2]).

The dissertation is organized as follows. In the first chapter we give a thorough overview of Heisenberg calculus, also called $\Psi_{\mathcal{V}}DO$ calculus, as presented in [BG] and [BGS]. For sake of clarity and completeness most of the proofs are given.

In the second chapter we develop a $\Psi_{\mathcal{V}}DO$ calculus with parameter in order to study the resolvent of an elliptic sublaplacian in the Heisenberg calculus. Here the methods Seeley [Se] and Shubin [Sh] cannot be extended to $\Psi_{\mathcal{V}}DO$ operators. So we use a new one built with almost homogeneous symbols with parameter. We construct then an algebra of $\Psi_{\mathcal{V}}DO$ with parameter in which the resolvent of an elliptic sublaplacian takes place as a parametrix (theorem 2.33). As immediate application we obtain that a selfadjoint elliptic sublaplacian is necessarily bounded from below.

In chapter 3 we introduce holomorphic families of $\Psi_{\mathcal{V}}DO$'s and we construct the complex powers of an elliptic sublaplacian. First in the case of a positive operator by means of the pseudodifferential construction of the heat kernel (theorem 3.17). Afterwards using the parametric $\Psi_{\mathcal{V}}DO$ calculus of chapter 2 when the operator is invertible or selfadjoint, but not necessarily positive (theorems 3.21 and 3.22).

In the fourth chapter we construct an analytic continuation of the trace on $\Psi_{\mathcal{V}}DO$'s with non-integral complex order as in [KV] and [CM2] we show that gives rise to a residual trace on $\Psi_{\mathcal{V}}DO$ with integral order which is an analogue of the non-commutative residue for $\Psi_{\mathcal{V}}DO$ operators (theorem 4.5 and proposition 4.9). Then we prove that this new non-commutative residue extends the Dixmier trace on the $\Psi_{\mathcal{V}}DO$ algebra (theorem 4.11) and is the unique trace up to a constant multiple on this algebra quotiented by the smoothing operators (theorem 4.15). As corollary we obtain a complete characterization of sums of commutators in the $\Psi_{\mathcal{V}}DO$ algebra(corollary 4.16).

In last chapter we give geometric applications of the non-commutative residue and the regularized trace. In the first section we define the zeta function of an elliptic sublaplacian and, in the selfadjoint case, we relate its residues and regular values to the coefficients of the heat kernel asymptotic (theorems 5.3 and 5.5). In section 5.2 we derive variational formulae for zeta functions with respect to C^1 families of sublaplacians. We use them in section 5.3 to produce conformal invariants associate to sublaplacians (theorem 5.14) extending then the results of N.K. Stanton [St].

In section 5.4 we look at the non-commutative geometry of pseudohermitian manifolds. In particular we are able to define the area of a compact three dimensional pseudohermitian manifold and to compute it by an explicit local formula involving the Tanaka-Webster scalar curvature (theorem 5.20).

In the last section we study the index of a square root of an elliptic sublaplacian. First we show that in even dimension the index is always zero and in odd dimension the index is given by the right coefficient of the heat kernel asymptotics (theorem 5.21).

Next using cyclic cohomology and the above local index formula of Connes-Moscovici we are able to show the existence of a de Rham's current whose pairing with the Chern character of a vector bundle gives the index with coefficients in the bundle and to give a local formula for this current as a universal finite linear combination of non-commutative residues of the kind of (125) (theorem 5.27).

Chapter 1

Hypoelliptic calculus on Heisenberg manifolds

In this chapter we shall give a thorough overview of Heisenberg calculus, also called $\Psi_{\mathcal{V}}DO$ calculus, following [BG] and [BGS] (see [Tay] and [EMM] for other point of views). For sake of clarity and completeness most of the proofs are given. In particular the heat kernel asymptotic is detailed.

1.1 Heisenberg manifolds

A *Heisenberg manifold* $(M, \mathcal{V})$ is a manifold M together with a hyperplane bundle $\mathcal{V} \subset TM$. Since they are different possible definitions for Heisenberg manifolds (e.g. [Ge3] and [EMM]) we stress the fact that here a Heisenberg manifold can be given either by an *integrable* or a *non-integrable* subbundle (*cf.* examples below).

A *Heisenberg diffeomorphism* $\phi : (M, \mathcal{V}) \to (M', \mathcal{V}')$ between two Heisenberg manifolds is a diffeomorphism from M onto M' such that $\phi_* \mathcal{V} = \mathcal{V}'$.

The local model for a (d+1)-dimensional Heisenberg manifold is an open subset U of $\mathbb{R}^{d+1}$ together with a hyperplane bundle $\mathcal{V} \subset TU$ and a $\mathcal{V}$*-frame* $X_0, X_1, \ldots, X_d$ of TU, i.e. $X_0, X_1, \ldots, X_d$ is a frame of TU and $X_1, \ldots, X_d$ generate $\mathcal{V}$. Then we define a *Heisenberg chart* for a Heisenberg manifold as a local Heisenberg diffeomorphism to such an open. There always exists locally a $\mathcal{V}$-frame, but globally this not true in general.

We have the following examples of Heisenberg manifolds:

- *Heisenberg group.* The 2-nilpotent group H_{2n+1} is associated to the Lie algebra generated by X_i, Y_i, $1 \leq i \leq n$, and T with relations $[X_i, Y_i] = T$ and the other brackets equal to 0. The Heisenberg structure is given by the hyperplane bundle generated by the $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$.

- *Codimension 1 foliations.* The foliation of M is given by an integrable hyperplane bundle which defines a Heisenberg structure on M.
- *Contact manifolds.* A contact manifold is a manifold M^{2n+1} together with a nowhere vanishing 1-form θ such that $d\theta$ is non degenerate on $\ker\theta$. The Heisenberg structure is given by $\mathcal{V} = \ker\theta$. A generic example of contact manifold is given by the cosphere bundle $S^*M = T^*M/\mathbb{R}_+^*$ of a manifold M. More precisely let $\omega = \sum dx_j \wedge d\xi_j$ the canonical symplectic form on T^*M and $R = \sum \xi_j d\xi_j$ the generator of the flow $\phi_s(x,\xi) = (x, e^s\xi)$ on T^*M. Then $\theta = \imath_R\omega = -\sum \xi_j dx_j$ is a contact form on S^*M.
- *Confoliations* [ET]. This is a mixed definition between contact manifolds and foliations. A confoliation structure on a manifold M^{2n+1} is given by a nowhere vanishing one form θ on M such that $(d\theta)^n \wedge \theta \geq 0$.
- *CR manifolds.* A CR manifold is an oriented manifold M^{2n+1} with a n-dimensional complex subbundle $T_{1,0}$ of the complexified tangent space $T_{\mathbb{C}}M$ such that $T_{1,0} \cap \bar{T_{1,0}} = \{0\}$. The Heisenberg structure is then given by $\mathcal{V} = \Re T_{1,0} \oplus \Re \bar{T}_{1,0}$. The basic example of a CR manifold is a real hypersurface M in $\mathbb{C}^n$ with the CR structure given by the maximal complex structure of M.
- *Pseudohermitian manifolds.* A pseudohermitian manifold is a CR manifold M^{2n+1} together with a contact form θ vanishing on $\mathcal{V} = \Re T_{1,0} \oplus \Re \bar{T}_{1,0}$ such that the Levi form of θ, i.e. the Hermitian form on $T_{1,0}$ given by

  $$L_\theta(V,W) = -d\theta(Z,\bar{W}), \tag{1.1}$$

  is positive definite. In this case there is a canonical connection associated to the pair (M,θ) called the Tanaka-Webster connection (see [Ta] and [We]).

The reason why the terminology of Heisenberg manifolds is used comes from the fact that we can at each point of a Heisenberg manifold attach a group isomorphic to a product $H_{2n+1} \times \mathbb{R}^{d-2n}$ (with the convention $H_1 = \{0\}$). The group structure is invariant by Heisenberg diffeomorphisms and comes with a family of dilations compatible with the group structure. So the structure involved is the structure of a Carnot group in the sense of [GrLP] and is just one step beyond vectorial spaces. For instance, in the case contact manifold M^{2n+1} we obtain the Heisenberg group H_{2n+1} at each point, whereas for a codimension 1 foliation on a manifold M^{d+1} we always get the abelian group $\mathbb{R}^{d+1}$.

Let us describe the tangent in the case of an open subset U of $\mathbb{R}^{d+1}$ together with a hyperplane bundle $\mathcal{V} \subset TU$ and with a $\mathcal{V}$-*frame* $X_0, X_1, \ldots, X_d$

of TU. Let $y \in U$ and let $x \to \varepsilon_y(x)$ the unique affine change which put y on the origin and such that the vector fields X_j coincides with $\frac{\partial}{\partial x_j}$ at $x = 0$ for $j = 0, 1, \ldots, d$. We call these new coordinates *y-coordinates*. In these coordinates the vector fields X_j have the form

$$X_j = \frac{\partial}{\partial x_j} + \sum_{j=0}^{d} \gamma_{jk}(x) \frac{\partial}{\partial x_k}, \qquad \gamma_{jk}(0) = 0. \tag{1.2}$$

Consider also the anisotropic dilations on $\mathbb{R}^{d+1}$ defined by

$$\lambda.x = \lambda.(x_0, x_1, \ldots, x_d) = (\lambda^2 x_0, \lambda x_1, \ldots, \lambda x_d), \qquad \lambda > 0. \tag{1.3}$$

We refer to these dilations as the *Heisenberg dilations*. For these dilations the vector fields $\frac{\partial}{\partial x_0}$ is homogeneous of degree -2, and $\frac{\partial}{\partial x_j}$, $1 \leq j \leq d$, is homogenous of degree -1.

The vector fields

$$X_0^y = \frac{\partial}{\partial x_0}, \tag{1.4}$$

$$X_j = \frac{\partial}{\partial x_j} + \sum_{j=0}^{d} c_{jk} x_k \frac{\partial}{\partial x_0}, \qquad 1 \leq j \leq d, \tag{1.5}$$

where $c_{jk} = c_{jk}(y) = \frac{\partial}{\partial x_k} \gamma_{jk}(0)$, are the homogeneous approximations of the vector fields X_j's. Indeed if we expand by means of Taylor expansions the coefficients of the vector fields X_j's we get

$$X_0 \sim X_0^y + X_0^{-1} + X_0^0 + \ldots, \tag{1.6}$$

$$X_j \sim X_j^y + X_j^0 + \ldots, \qquad 1 \leq j \leq d, \tag{1.7}$$

with X_j^k homogenous of degree k with respect to the Heisenberg dilations.

The vector fields $X_0^y, \ldots, X_d^y$ generate a 2-step nilpotent Lie algebra. They are then left-invariant vector fields over a 2-step nilpotent Lie group with underlying space $\mathbb{R}^{d+1}$ and product given by

$$(x.z)_0 = x_0 + y_0 + \frac{1}{2} \sum_{j,k=1}^{d} c_{jk} x_j z_k, \tag{1.8}$$

$$(x.z)_j = x_j + y_j, \qquad 1 \leq j \leq d. \tag{1.9}$$

This group is called the *y-group*.

It is actually possible to construct the y-group in terms of Gromov-Hausdorff limits of metric spaces, so that the y-group is tangent to the manifold at y (see [Gr] and [Be]). We can also see it as the boundary of a tangent groupoid as in [Co1] using the Heisenberg dilations (1.3) for blowing up the diagonal of $M \times M$.

1.2 Sublaplacians and ideas of the $\Psi_{\mathcal{V}}DO$ calculus

Let $(M^{d+1}, \mathcal{V})$ be a Heisenberg manifold. A *sublaplacian* on M is a differential operator which can locally be written as

$$\Delta = -\sum_{j=1}^{d} X_j^2 - i\lambda(x)X_0 + \sum_{j=1}^{d} \mu_j(x)X_j + \nu(x), \tag{1.10}$$

where $\lambda, \mu_1, \ldots, \mu_d, \nu$ are smooth functions and $X_0, X_1, \ldots, X_d$ is a local $\mathcal{V}$-frame of TM.

We can also define sublaplacians acting on the sections of a vector bundle $\mathcal{E}$ over M as follows: they are differential operators acting on the sections of $\mathcal{E}$ which are locally of the form

$$\Delta = -\sum_{j=1}^{d} X_j^2 \otimes \mathrm{id}_{\mathcal{E}} - i\lambda(x)X_0 + \sum_{j=1}^{d} \mu_j(x)X_j + \nu(x), \tag{1.11}$$

where $X_0, X_1, \ldots, X_d$ is a local $\mathcal{V}$-frame of TM, λ is a smooth function with diagonal matrices values and $\mu_1, \ldots, \mu_d, \nu$ are smooth functions with matrices values.

This kind of operator cannot be elliptic. Nevertheless it can be hypoelliptic and in this case the $\Psi_{\mathcal{V}}DO$ calculus allows us to construct a parametrix. The basic idea is first, using the dilations (1.3), to consider

$$\Delta_2 = -\sum_{j=1}^{d} X_j^2 - i\lambda(x)X_0, \tag{1.12}$$

as having order 2. Then developping a suitable symbolic calculus we need only to find a parametrix for Δ_2. Second, we freeze the coefficients of Δ_2 by modelizing it at each point y of U by the y-invariant differential operator

$$\Delta_2^y = -\sum_{j=1}^{d} (X_j^y)^2 - i\lambda X_0^y, \qquad \lambda = \lambda(y). \tag{1.13}$$

Under some condition on the function λ the operator Δ_2^y is invertible for any y and the inverses give the principal symbol of a parametrix for Δ (*cf.* section 1.8).

Moreover using a partition of unity and elementary algebra one can construct a sublaplacian on any Heisenberg manifold and we can do this in such way the operator is selfadjoint and admits a parametrix (i.e. is elliptic in the Heisenberg calculus).

Let us now give some examples of sublaplacians on a a pseudohermitian manifold (M^{2n+1}, θ). The CR structure is defined by a complex n-dimensional subbundle of $T_{\mathbb{C}}M$ such that

$$T_{1,0} \cap T_{0,1} = \{0\}, \qquad T_{0,1} = \bar{T}_{1,0}. \tag{1.14}$$

The Levi form is the positive definite Hermitian form on $T_{1,0}$ defined by

$$L_\theta(Z,W) = -id\theta(Z,\bar{W}). \tag{1.15}$$

There is an unique vector fields X on M such that

$$\theta(X) = 1, \qquad \iota_X d\theta = 0. \tag{1.16}$$

The contact form θ determines a Hermitian metric on M by

$$X \perp T_{1,0} \qquad \text{and} \qquad |X| = 1; \tag{1.17}$$

$$T_{1,0} \perp T_{0,1} \qquad \text{and} \qquad \text{complex conjugation is an isometry;} \tag{1.18}$$

$$\langle Z,W\rangle = L_\theta(Z,W) \qquad \text{for } Z,W \in T_{1,0}. \tag{1.19}$$

This Hermitian metric defines by duality a Hermitian metric on forms and together with the volume form $(d\theta)^n \wedge \theta$ endows the forms with an inner product.

The *Kohn complex* [Ko] is realized as follows. The bundle of covectors $(1,0)$ is

$$\Lambda^{1,0} = \{\text{annihilator of } T_{1,0} \oplus \mathbb{C}X\} \subset T^*_{\mathbb{C}}M. \tag{1.20}$$

Similarly,

$$\Lambda^{0,1} = \{\text{annihilator of } T_{0,1} \oplus \mathbb{C}X\} \subset T^*_{\mathbb{C}}M. \tag{1.21}$$

The bundle of covectors (p,q) is

$$\Lambda^{p,q} = (\Lambda^{1,0})^p \wedge (\Lambda^{0,1})^q \subset \Lambda^{p+q}T^*_{\mathbb{C}}M. \tag{1.22}$$

A (p,q)-form is a section of $\Lambda^{p,q}$ and the space of (p,q)-forms is denoted $\mathcal{E}^{p,q}$. Set

$$\bar\partial_{b,q} : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p,q+1}, \qquad \bar\partial_{b,q} = \pi_{p,q+1} \circ d, \tag{1.23}$$

where d is the exterior derivative and $\pi_{p,q+1}$ is the orthogonal projection onto $\mathcal{E}^{p,q+1}$. Then

$$\bar\partial_b : \mathcal{E}^p = \bigoplus \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^p, \tag{1.24}$$

is a chain complex. The *Kohn Laplacian* on (p,q)-forms is then

$$\Box_{b,q} = \bar\partial^*_{b,q}\bar\partial_{b,q} + \bar\partial_{b,q+1}\bar\partial^*_{b,q+1}, \tag{1.25}$$

and one can show it is a sublaplacian acting on complex forms (see [BG]).

We can similarly defines a *real* sublaplacian. Let H^* be the orthogonal complement of θ in T^*M so that H^* is the dual of the maximal complex tangent space $H(M)$ and let π be the orthogonal projection onto H^*. Set

$$d_b = \pi \circ d : C^\infty(M) \to H^*. \tag{1.26}$$

The *pseudohermitian sublaplacian* [Le] is defined as

$$\Delta_b = d_b^* d_b. \tag{1.27}$$

In [Le] it is shown that $\Delta_b = 2\Re\Box_b$, where $\Box_b$ is the Kohn Laplacian on functions. So Δ_b is also a sublaplacian.

However, the operator Δ_b does not transform conformally under conformal changes of the contact form θ. So, in order to study the Yamabe problem on CR manifolds, Jerison and Lee [JL1] introduced the operator

$$\boxdot_\theta = \Delta_b + \frac{n}{n+2} R_n, \tag{1.28}$$

where R_n is the scalar curvature of the Tanaka-Webster connection. Then

$$\boxdot_{e^{2f}\theta} = e^{-(n+2)f}\, \boxdot_\theta\, e^{nf}, \qquad f \in C^\infty(M), \tag{1.29}$$

so we shall call $\boxdot_\theta$ the *conformal (pseudohermitian) sublaplacian.*

1.3 Classes of Heisenberg symbols

From now on U is an open subset of $\mathbb{R}^{d+1}$ together with a hyperplane bundle $\mathcal{V} \subset TU$ and a $\mathcal{V}$-frame $X_0, X_1, \ldots, X_d$ of TU. For $x \in U$ we denote by ε_x the affine map onto the x-coordinates. This the unique affine change of coordinates which put x at the origin and such that the vector fields X_j coincides with $\frac{\partial}{\partial_{x_j}}$ at x. Also we set

$$\sigma(x,\xi) = (\sigma_0(x,\xi), \sigma_1(x,\xi), \ldots, \sigma_d(x,\xi)), \tag{1.30}$$

where $\sigma_j(x,\xi)$ is the symbol of $\frac{1}{i}X_j$. We refer to σ as the (real) symbol of the frame $X_0, X_1, \ldots, X_d$.

In this section we define the convenient symbols for Heisenberg manifolds. At a local level they are associate to the anisotropic Heisenberg dilations

$$\lambda.\xi = (\lambda^2\xi_0, \lambda\xi_1, \ldots, \lambda\xi_d), \qquad \lambda > 0, \tag{1.31}$$

and to the homogeneous pseudo-norm defined by

$$\|\xi\| = (|\xi_0|^2 + |\xi_1|^4 + \ldots + |\xi_d|^4)^{\frac{1}{4}}, \qquad \xi \in \mathbb{R}^{d+1}. \tag{1.32}$$

Definition 1.1 *$S_m(\mathbb{R}^{d+1})$, $m \in \mathbb{C}$, is the space of functions $f \in C^\infty(\mathbb{R}^{d+1}\setminus 0)$ which are homogeneous of degree m with respect to the Heisenberg dilations, that is*

$$f(\lambda.\xi) = \lambda^m f(\xi), \qquad \lambda > 0. \tag{1.33}$$

The homogeneity implies that f satisfies to the estimates

$$|\partial_\xi^\alpha f(\xi)| \leq C_\alpha \|\xi\|^{\Re m - \langle\alpha\rangle}, \qquad \xi \neq 0, \tag{1.34}$$

where we have let $\langle\alpha\rangle = \alpha_0 + |\alpha| = 2\alpha_0 + \alpha_1 + \ldots + \alpha_d$. Therefore a smoothed version of f belongs to the following class of symbols

Definition 1.2 *$S_{||}^k(\mathbb{R}^{d+1})$, $k \in \mathbb{R}$, is the Fréchet space of functions $f \in C^\infty(\mathbb{R}^{d+1})$ satisfying to the estimates*

$$|\partial_\xi^\alpha f(\xi)| \leq C_\alpha (1 + \|\xi\|)^{k - \langle\alpha\rangle}. \tag{1.35}$$

Its topology is defined by means of the semi-norms given by the lower bounds in the estimates (1.35).

Definition 1.3 *$S^m(\mathbb{R}^{d+1})$, $m \in \mathbb{C}$, is the space of functions $f \in C^\infty(\mathbb{R}^{d+1})$ with an asymptotic expansion*

$$f(\xi) \sim \sum_{j \geq 0} f_{m-j}(\xi), \qquad f_k \in S_k(\mathbb{R}^{d+1}), \tag{1.36}$$

in the sense that for any integer N we have

$$|\partial_\xi^\alpha (f - \sum_{j<N} f_{m-j})(x,\xi)| \leq C_{\alpha N} \|\xi\|^{\Re m - \langle\beta\rangle - N}, \qquad \|\xi\| \geq 1. \tag{1.37}$$

We can define smooth families versions of the previous classes of symbols as follows.

Definition 1.4 *$S_m(U \times \mathbb{R}^{d+1})$, $m \in \mathbb{C}$, is the space of functions $f \in C^\infty(U \times \mathbb{R}^{d+1}\setminus 0)$ which are homogeneous of degree m in the last variable, i.e.*

$$f(x, \lambda.\xi) = \lambda^m f(x,\xi), \qquad \lambda > 0. \tag{1.38}$$

Definition 1.5 *$S_{||}^k(U \times \mathbb{R}^{d+1})$, $k \in \mathbb{R}$, is the space of functions $f \in C^\infty(U \times \mathbb{R}^{d+1})$ satisfying to the estimates*

$$|\partial_x^\alpha \partial_\xi^\beta f(\xi)| \leq C_{\alpha\beta}(x)(1 + \|\xi\|)^{k - \langle\beta\rangle}, \tag{1.39}$$

with $C_{\alpha\beta}(x)$ locally bounded on U.

Definition 1.6 $S^m(U \times \mathbb{R}^{d+1})$, *$m \in \mathbb{C}$, is the space of functions $f \in C^\infty(U \times \mathbb{R}^{d+1})$ with an asymptotic expansion*

$$f(x,\xi) \sim \sum_{j\geq 0} f_{m-j}(x,\xi), \qquad f_k \in S_k(U \times \mathbb{R}^{d+1}), \tag{1.40}$$

in the sense that for any integer N we have

$$|\partial_x^\alpha \partial_\xi^\beta (f - \sum_{j<N} f_{m-j})(x,\xi)| \leq C_{\alpha\beta N}(x) \|\xi\|^{\Re m - \langle\alpha\rangle - N}, \qquad \|\xi\| \geq 1. \tag{1.41}$$

where $C_{\alpha\beta NJ}(x)$ is a locally bounded function on U.

Remark 1.7 As $a(1+|\xi|)^{\frac{1}{2}} \leq 1 + \|\xi\| \leq b(1+|\xi|)$ it follows from [Ho1, theorem 2.9] that the asymptotic expansion (1.40) is equivalent to the requirement of the following two conditions:

(i) For any multi-orders α and β there exists a real $\mu_{\alpha\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta f(x,\xi)| \leq C_{\alpha\beta}(x)(1+|\xi|)^{\mu_{\alpha\beta}}, \tag{1.42}$$

with $C_{\alpha\beta}(x)$ locally bounded.

(ii) For any integer N if J is large enough we have

$$|f(x,\xi) - \sum_{j\leq J} f_{m-j}(x,\xi)| \leq C_{NJ}(x)|\xi|^{-N}, \qquad \|\xi\| \geq 1, \tag{1.43}$$

with $C_{NJ}(x)$ locally bounded.

In particular the asymptotic expansion for Heisenberg symbols is the same as usual.

Proposition 1.8 ([BG]) *Let $m \in \mathbb{C}$ and suppose given for $j = 0, 1, \ldots$ some symbol f_{m-j} in $S_{m-j}(U \times \mathbb{R}^{d+1})$. Then there exists $f \in S^m(U \times \mathbb{R}^{d+1})$ such that $f \sim \sum f_{m-j}$. Moreover f is unique modulo $S^{-\infty}(U \times \mathbb{R}^{d+1}) = \cap S^k_{||}(U \times \mathbb{R}^{d+1})$.*

Let us now define almost-homogeneous symbols.

Definition 1.9 $S^m_{ah}(U \times \mathbb{R}^{d+1})$, *$m \in \mathbb{C}$, is the space of functions $f \in C^\infty(U \times \mathbb{R}^{d+1})$ almost homogeneous of degree m in the last variable, i.e.*

$$f(x, \lambda.\xi) - \lambda^m f(x,\xi) \in S^{-\infty}(U \times \mathbb{R}^{d+1}), \qquad \lambda > 0. \tag{1.44}$$

The interest of the definition relies on the proposition:

Proposition 1.10 ([BG]) *Let $f \in C^\infty(U \times \mathbb{R}^{d+1})$. Then the following are equivalent:*

(i) f belongs to $S^m_{ah}(U \times \mathbb{R}^{d+1})$.

(ii) f lies in $S^m(U \times \mathbb{R}^{d+1})$ and has only one term in its asymptotic expansion, i.e. $f \sim f_m$ with $f_m \in S_m(U \times \mathbb{R}^{d+1})$.

Proof. If f belongs to $S^m(U\times\mathbb{R}^{d+1})$ and has only one term in its asymptotic expansion then f is almost homogeneous of degree m.

Conversely, suppose that f is in $S^m_{ah}(U \times \mathbb{R}^{d+1})$. Then we have

$$|f(x,t.\xi) - t^m f(x,\xi)| \leq C_{tN}(x)(1 + \|\xi\|)^{-N}, \tag{1.45}$$

with $C_{tN}(x)$ locally bounded. If we replace ξ by $s.\xi$, $s > 0$, and if N is large enough we get

$$|s^m f(x,st.\xi) - t^m f(x,s.\xi)| \leq C_{tN}(x)s^{m-1}\|\xi\|^{-N}, \qquad \xi \neq 0. \tag{1.46}$$

Define now the sequence $(g_k)_{k\geq 0} \subset C^\infty(U \times (\mathbb{R}^{d+1}\backslash 0))$ by

$$g_k(x,\xi) = (2^k)^{-m} f(x,2^k.\xi), \qquad \xi \neq 0. \tag{1.47}$$

Then by (1.46) we have

$$|g_{k+1}(x,\xi) - g_k(x,\xi)| \leq C_{2N}(x)2^{-k}\|\xi\|^{-N}, \qquad \xi \neq 0. \tag{1.48}$$

As there are estimates similar to (1.45) for the derivatives of $f(x,t.\xi) - t^m f(x,\xi)$, the series $\sum_{k\geq 0}(g_{k+1} - g_k)$ is convergent in $C^\infty(U \times (\mathbb{R}^{d+1}\backslash 0))$. This implies that the sequence (g_k) converges in $C^\infty(U \times (\mathbb{R}^{d+1}\backslash 0))$ to some $g \in C^\infty(U \times (\mathbb{R}^{d+1}\backslash 0))$ such that for any N we have

$$|g(x,\xi) - f(x,\xi)| \leq C_{2N}(x)\|\xi\|^{-N}, \qquad \xi \neq 0. \tag{1.49}$$

If we take $s = 2^k$ in (1.46) and let $k \to \infty$ with t fixed we get the required homogeneity of g. Moreover with $k = 0$ the estimate (1.49) and similar estimates for the derivatives show that f lies in $S^m(U \times \mathbb{R}^{d+1})$ and $f \sim g$. ■

1.4 The $\Psi_{\mathcal{V}}DO$ operators on an open subset of $\mathbb{R}^{d+1}$

The $\Psi_{\mathcal{V}}DO$ calculus is the pseudodifferential calculus associated to the quantization of $S^*(U \times \mathbb{R}^{d+1})$ given by the map

$$f \longrightarrow f(x,\sigma(x,D)). \tag{1.50}$$

Here $f(x,\sigma(x,D))$ denotes the continuous operator from $C^\infty_c(U)$ into $C^\infty(U)$ defined by

(1.51)

$$f(x,\sigma(x,D))u(x) = (2\pi)^{-(d+1)} \int e^{ix.\xi} f(x,\sigma(x,\xi))\hat{u}(\xi)d\xi, \qquad u \in C^\infty_c(U).$$

More precisely:

Definition 1.11 *A $\Psi_{\mathcal{V}}DO$ operator of order m, $m \in \mathbb{C}$, is a continuous operator from $C_c^\infty(U)$ into $C^\infty(U)$ of the form*

$$P = f(x, \sigma(x, D)) + R, \tag{1.52}$$

with $f \in S^m(U \times \mathbb{R}^{d+1})$, called the symbol of P, and R smoothing operator. The space of $\Psi_{\mathcal{V}}DO$'s of order m is denoted $\Psi_{\mathcal{V}}^m(U)$.

Proposition 1.12 ([BG]) *Let $m \in \mathbb{C}$. Then:*

1) *The space $\Psi_{\mathcal{V}}^m(U)$ does not depend on the choice of the $\mathcal{V}$-frame $X_0, X_1, \ldots, X_d$.*

2) *Any $P \in \Psi_{\mathcal{V}}^m(U)$ has a kernel smooth outside the diagonal and extends to a continuous linear mapping from $\mathcal{E}'(U)$ into $\mathcal{D}'(U)$. It is a smoothing operator if, and only if, its symbol belongs to $S^{-\infty}(U \times \mathbb{R}^{d+1})$.*

3) *Set $k = \Re m$ if $\Re m \geq 0$ and $k = \frac{1}{2}\Re m$ otherwise. We have*

$$\Psi_{\mathcal{V}}^m(U) \subset \Psi_{\frac{1}{2},\frac{1}{2}}^k(U) \tag{1.53}$$

where $\Psi_{\frac{1}{2},\frac{1}{2}}^k(U)$ is the space of pseudodifferential operators of type $(\frac{1}{2}, \frac{1}{2})$ (see [Ho1]).

Combining the last inclusion with the Calderón-Vaillancourt theorem ([CV], [Hw]) we obtain the Sobolev regularity of $\Psi_{\mathcal{V}}DO$ operators.

Proposition 1.13 *Let $P \in \Psi_{\mathcal{V}}^m(U)$ and set $k = \Re m$ if $\Re m \geq 0$ and $k = \frac{1}{2}\Re m$ otherwise. Then for any real s the operator P extends to a continuous linear mapping*

$$P : H^s_{\text{comp}}(U) \longrightarrow H^{s-k}_{\text{loc}}(U). \tag{1.54}$$

Recall that an operator P given by a kernel $k_P(x, y) \in \mathcal{D}'(U \times U)$ is said *properly supported* if both projections $\pi_x, \pi_y : \operatorname{supp} k_P(x, y) \to U$ are proper maps.

Proposition 1.14 *Let $m \in \mathbb{C}$. Then:*

1) *Each $\Psi_{\mathcal{V}}DO$ operator P can be written as $P = Q + R$ with Q properly supported $\Psi_{\mathcal{V}}DO$ and R smoothing operator.*

2) *If P is a properly supported $\Psi_{\mathcal{V}}DO$ operator, then it induces a continuous endomorphism of $C_c^\infty(U)$ and extends to continuous endomorphisms of $C^\infty(U)$, $\mathcal{E}'(U)$ and $\mathcal{D}'(U)$ respectively.*

1.5 Composition of $\Psi_{\mathcal{V}}DO$ operators

Let us look as the composition of $\Psi_{\mathcal{V}}DO$ operators. As $\Psi^*_{\mathcal{V}}(U)$ is contained in $\Psi_{\frac{1}{2},\frac{1}{2}}(U)$ the classical asymptotic expansion

$$q_1\#q_2(x,\xi)\sum\frac{1}{\alpha!}D_\xi q_1(x,\xi)\partial_x q_2(x,\xi), \tag{1.55}$$

for the symbol of the product of two pseudodifferential operators does not make sense. However, it is possible to show that the composition of two $\Psi_{\mathcal{V}}DO$'s can be a $\Psi_{\mathcal{V}}DO$ and then to derive an asymptotic formula for the symbol of the product. But instead of involving the pointwise product of symbols it is given in terms of Fourier convolution of homogeneous symbols with respect to the tangent group structure at each point.

Let us first define precisely the convolution for symbols. To this end for any $y\in U$ we denote by X_j^y the left-invariant vector fields with respect to the y-group which coincides with $\frac{\partial}{\partial x_j}$ at the origin. Then we let $\sigma^y(x,\xi)$ be the symbol of the frame $X_0^y, X_1^y,\ldots,X_d^y$.

Lemma 1.15 ([BG]) *Let $y\in U$. Then:*

1) For any $f\in S_{||}^k(\mathbb{R}^{d+1})$ the operator $f(\sigma^y(x,D))$ maps $\mathcal{S}(\mathbb{R}^{d+1})$ to itself.

2) For $f_1\in S_{||}^{k_1}(\mathbb{R}^{d+1})$ and $f_2\in S_{||}^{k_2}(\mathbb{R}^{d+1})$ we have

$$f_1(\sigma^y(x,D))\circ f_2(\sigma^y(x,D)) = (f_1 *^y f_2)(\sigma^y(x,D)), \tag{1.56}$$

where $y\to *^y$ is a smooth family of continuous bilinear maps from $S_{||}^{k_1}(\mathbb{R}^{d+1})\times S_{||}^{k_2}(\mathbb{R}^{d+1})$ into $S_{||}^{k_1+k_2}(\mathbb{R}^{d+1})$.

Remark 1.16 If $X_j^y=\frac{\partial}{\partial x_j}$, $j=0,1,\ldots,d$, then $\sigma^y(x,\xi)=\xi$ and we have

$$f_1(\sigma^y(x,D))\circ f_2(\sigma^y(x,D)) = f_1(D)\circ f_2(D) = (f_1f_2)(D). \tag{1.57}$$

So in this case the convolution $*^y$ reduces to the pointwise product of symbols.

Remark 1.17 In general the convolution symbol $f_1 *^y f_2$ is given by

$$f_1 *^y f_2(x,\xi) = (2\pi)^{-(d+1)}\iint e^{-i\langle z,\eta\rangle} f_1(\xi+\eta) f_2(\sigma^y(z,\xi))dzd\eta, \tag{1.58}$$

where the integral is taken in the sense of oscillating integrals. Indeed there exists a differential operator $L=L(\xi,z,\eta,D_z,D_\eta)$, independent of y, such that

$$L^t(e^{-i\langle z,\eta\rangle}) = e^{-i\langle z,\eta\rangle}, \tag{1.59}$$

and for N large enough the integral

$$(2\pi)^{-(d+1)} \iint e^{-i\langle z,\eta\rangle} L^N(f_1(\xi+\eta)f_2(\sigma^y(z,\xi)))dzd\eta, \tag{1.60}$$

is absolutely convergent.

As $y \to *^y$ is a smooth family we get a continuous bilinear map

$$*: S_{||}^{k_1}(U\times\mathbb{R}^{d+1}) \times S_{||}^{k_2}(U\times\mathbb{R}^{d+1}) \longrightarrow S_{||}^{k_1+k_2}(U\times\mathbb{R}^{d+1}), \tag{1.61}$$

by setting

$$f_1 * f_2(y,\xi) = (f_1(y,.) *^y f_2(y,.))(\xi), \qquad f_j \in S_{||}^{k_j}(U\times\mathbb{R}^{d+1}). \tag{1.62}$$

This applies in particular if f_1 and f_2 are almost homogeneous symbols of degree m_1 and m_2. But for any $t>0$ we have

$$\begin{aligned} &f_1 * f_2(x,t.\xi) - t^{m_1+m_2} f_1 * f_2(x,\xi) = \\ &\quad (f_1(x,t.\xi) - t^{m_1} f_1(x,\xi)) * f_2(x,t.\xi) + t^{m_1} f_1(x,\xi) * (f_2(x,t.\xi) - t^{m_2} f_2(x,\xi)). \end{aligned} \tag{1.63}$$

So using the fact that $S^{-\infty}(U\times\mathbb{R}^{d+1})$ is a two-side ideal for $*$ we see that $f_1 * f_2$ is almost-homogeneous of degree m_1+m_2 and its principal symbol depends only on the principal symbol of f_1 and f_2. Then identifying $S_m(U\times\mathbb{R}^{d+1})$ with the quotient $S_{ah}^m(U\times\mathbb{R}^{d+1})/S^{-\infty}(U\times\mathbb{R}^{d+1})$ we obtain a bilinear map

$$*: S_{m_1}(U\times\mathbb{R}^{d+1}) \times S_{m_2}(U\times\mathbb{R}^{d+1}) \longrightarrow S_{m_1+m_2}(U\times\mathbb{R}^{d+1}). \tag{1.64}$$

This gives the needed convolution for homogeneous symbols.

We can now state the composition formula for $\Psi_{\mathcal{V}}DO$ operators using the notations which follow. If $f(x,\xi)$ is a symbol let

$$f_\alpha^{\beta\gamma}(x,\xi) = \xi^\gamma \partial_x^\alpha \partial_\xi^\beta f(x,\xi) \qquad \text{and} \qquad f^\delta(x,\xi) = D_\xi^\delta f(x,\xi). \tag{1.65}$$

Next denote by $\sigma^{(x)}(z,\xi) = (\varepsilon_x)_*\sigma(z,\xi)$ the $\mathcal{V}$-frame symbol in the x-coordinates and set

$$h_{\alpha\beta\gamma\delta}(x) = \frac{1}{\delta!}\partial_z^\delta h_{\alpha\beta\gamma}(x,0), \qquad h_{\alpha\beta\gamma}(x,z) = \frac{1}{\alpha!\beta!}(\varepsilon_x'^{-1}(z))^\alpha e_{\beta\gamma}(x,z), \tag{1.66}$$

where the functions $e_{\beta\gamma}$ are defined by the equality

$$(\sigma^{(x)}(z,\xi) - \sigma^x(z,\xi))^\beta = \sum_{|\gamma|=|\beta|} e_{\beta\gamma}(x,z)\sigma^x(z,\xi)^\gamma. \tag{1.67}$$

Proposition 1.18 ([BG]) *Let $P_1 \in \Psi_{\mathcal{V}}^{m_1}(U)$ with symbol $f_1 \sim \sum f_{1,m_1-j}$ and $P_2 \in \Psi_{\mathcal{V}}^{m_2}(U)$ with symbol $f_2 \sim \sum f_{2,m_2-j}$ and suppose either Q_1 or Q_2 properly supported. Then P_1P_2 is a $\Psi_{\mathcal{V}}DO$ of order m_1+m_2 and has symbol $f \sim \sum f_{m_1+m_2-j}$ with*

$$f_{m_1+m_2-j}(x,\xi) = \sum h_{\alpha\beta\gamma\delta}(x) f_{1,m_1-k}^{\delta} * f_{2,m_2-l,\alpha}^{\beta\gamma}(x,\xi), \tag{1.68}$$

where the summation is taken over the indices such that

$$|\gamma| = |\beta|, \qquad |\beta| + |\alpha| \leq \langle\delta\rangle + \langle\beta\rangle - \langle\gamma\rangle = j-k-l. \tag{1.69}$$

In particular, the principal symbol of P_1P_2 is $f_{1,m_1} * f_{2,m_2}(x,\xi)$ is the convolution of the principal symbols of P_1 and P_2.

By proposition 1.14 the composition P_1P_2 is well defined as a continuous mapping from $C_c^\infty(U)$ into $C^\infty(U)$ and up to a smoothing operator we have

$$P_1P_2 = \sum \varphi_i f_1(x,\sigma(x,D))\psi_i f_2(x,\sigma(x,D)), \tag{1.70}$$

where (φ_i) and (ψ_i) are families of smooth compactly supported functions on U such that (φ_i) is a locally finite partition of unity and $\psi_i = 1$ near $\operatorname{supp}\varphi_i$. So the proposition follows from the lemma:

Lemma 1.19 ([BG]) *Let $\psi \in C_c^\infty(U)$ and let $f_1 \in S_{||}^{k_1}(U \times \mathbb{R}^{d+1})$ and $f_2 \in S_{||}^{k_2}(U \times \mathbb{R}^{d+1})$. Then*

$$f_1(x,\sigma(x,D))\psi f_2(x,\sigma(x,D)) = f_1 \#_\psi f_2(x,\sigma(x,D)), \tag{1.71}$$

with $f_1\#_\psi f_2$ in $S_{||}^{m_1+m_2}(U\times\mathbb{R}^{d+1})$ such that

$$f_1\#_\psi f_2(x,\xi) \sim \sum h_{\alpha\beta\gamma\delta}(x)\psi(x) f_1^{\delta} * (\psi f)_{2,\alpha}^{\beta\gamma}(x,\xi), \tag{1.72}$$

where the summation is taken over the indices such that $|\gamma| = |\beta|$ and $|\beta| + |\alpha| = \langle\delta\rangle + \langle\beta\rangle - \langle\gamma\rangle$.

Remark 1.20 In the expansion (1.72) the symbol $f_1^\delta * (\psi f)_{2,\alpha}^{\beta\gamma}$ has order $k_1 + k_2 - \langle\delta\rangle - \langle\beta\rangle + \langle\gamma\rangle$. So there are only finitely many terms of a given order and this asymptotic expansion does make sense.

Remark 1.21 Suppose $\mathcal{V}$ is the trivial hyperplane bundle generated by $\frac{\partial}{\partial x_1}$, ..., $\frac{\partial}{\partial x_d}$. Then the convolution $*$ is the pointwise product for functions and we have $\varepsilon_x(y) = y - x$ and $\sigma^{(x)}(z,\xi) = \sigma^x(z,\xi) = \xi$. This implies that $h_{\alpha 00\alpha} = \frac{1}{\alpha!}$ and $h_{\alpha\beta\gamma\delta} = 0$ if $(\beta,\gamma,\delta) \neq (0,0,\alpha)$. Thus

$$f_1\#_\psi f_2(x,\xi) \sim \sum \frac{1}{\alpha!} f_1^\alpha * (\psi f_{2,\alpha})^{00}(x,\xi) \sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha f_1(x,\xi) D_x^\alpha(\psi f_2)(x,\xi), \tag{1.73}$$

which gives back the asymptotic expansion for the symbol of the product of two standard pseudo-differential operators.

Remark 1.22 The symbol $f_1 \#_\psi f_2$ is given by the oscillating integral
(1.74)
$$f_1 \#_\psi f_2(x,\xi) = (2\pi)^{-(d+1)} \iint e^{-i\langle z,\eta\rangle} f_1(x,\xi+\eta)\psi(z) f_2(\varepsilon_x^{-1}(z), \sigma^{(x)}(z,\xi)) dz d\eta.$$

This integral can be regularized as the integral (1.58) and by taking suitable Taylor expansions inside one can obtain the asymptotic expansion (1.72). In fact, for N integer let $\Sigma^{(N)}$ the summation over the indices such that $|\gamma| = |\beta|$ and $|\beta| + |\alpha| = \langle\!\langle\delta\rangle\!\rangle + \langle\!\langle\beta\rangle\!\rangle - \langle\!\langle\gamma\rangle\!\rangle < N$. Then it follows from the closed graph theorem that the bilinear map

$$(f_1, f_2) \longrightarrow f_1 \#_\psi f_2 - \sum^{(N)} h_{\alpha\beta\gamma\delta}(x)\psi(x) f_1^\delta * (\psi f)_{2,\alpha}^{\beta\gamma}(x,\xi), \tag{1.75}$$

is continuous from $S_{||}^{k_1}(U \times \mathbb{R}^{d+1}) \times S_{||}^{k_2}(U \times \mathbb{R}^{d+1})$ into $S_{||}^{k_1+k_2-N}(U \times \mathbb{R}^{d+1})$.

1.6 Kernels of $\Psi_\mathcal{V} DO$ operators

Let $K(x) \in \mathcal{S}'(\mathbb{R}^{d+1})$. For $\lambda > 0$ we denote by $K(\lambda.x)$ the distribution defined by

$$\langle K(\lambda.x), u(x)\rangle = \lambda^{-(d+2)} \langle K(x), u(\lambda^{-1}.x)\rangle, \qquad u \in \mathcal{S}(\mathbb{R}^{d+1}). \tag{1.76}$$

Then K is said to be homogeneous of degree m, $m \in \mathbb{C}$, if

$$K(\lambda.x) = \lambda^m K(x) \qquad \text{for any } \lambda > 0. \tag{1.77}$$

The starting point here is the problem of the extension of a homogeneous symbol on $\mathbb{R}^{d+1}\backslash 0$ into a homogeneous distribution on $\mathbb{R}^{d+1}$.

Lemma 1.23 ([BG]) *Let $f \in C^\infty(\mathbb{R}^{d+1}\backslash 0)$ homogeneous of degree m.*

1) If m is not an integer $\leq -(d+2)$, then f has an unique homogeneous extension as a tempered distribution on $\mathbb{R}^{d+1}$.

2) If m is an integer $\leq -(d+2)$, the only obstructions to such an extension are given by the non vanishing of

$$c_\alpha(f) = \frac{1}{\alpha!} \int_{||\xi||=1} \xi^\alpha f(\xi) i_E d\xi, \qquad \langle\!\langle\alpha\rangle\!\rangle = -(m+d+2), \tag{1.78}$$

where E is the generator of the flow $\phi_s(\xi) = (e^{2s}\xi_0, e^s\xi')$.

Proof. If $\Re m > -(d+2)$ then f is integrable near the origin and defines a tempered distribution which is its unique homogeneous extension.

If $\Re m \leq -(d+2)$ an extension of f as a tempered distribution on $\mathbb{R}^{d+1}$ is provided by

(1.79)
$$\langle \tau(\xi), u(\xi) \rangle = \int (u(\xi) - \psi(\|\xi\|) \sum_{\langle\alpha\rangle \leq k} \frac{\xi^\alpha}{\alpha!} u^{(\alpha)}(0)) f(\xi) d\xi, \qquad u \in \mathcal{S}(\mathbb{R}^{d+1}),$$

with $k \geq -(\Re m + d + 2)$ and $\psi \in C_c^\infty(\mathbb{R}_+)$ such that $\psi = 1$ near 0. One can check that

$$\tau(\lambda.\xi) - \lambda^m \tau(\xi) = \lambda^m \sum_{\langle\alpha\rangle \leq k} \rho_\alpha(\lambda) \frac{1}{\alpha!} \int_{\|\xi\|=1} \xi^\alpha f(\xi) i_E d\xi, \qquad \lambda > 0, \tag{1.80}$$

with $\rho_\alpha(\lambda) = \int_0^\infty \mu^{\langle\alpha\rangle + m + d + 2} (\psi(\mu) - \psi(\lambda\mu)) \frac{d\mu}{\mu}$. Let $\lambda = e^s$ and write ψ in the form

$$\psi(\mu) = h(\log \mu) \tag{1.81}$$

with $h \in C^\infty(\mathbb{R})$ such that $h = 1$ near $-\infty$ and $h = 0$ near $+\infty$. Then

$$\frac{d}{ds} \rho_\alpha(e^s) = -e^{-as} \int_{-\infty}^{+\infty} e^{at} h'(t) dt, \qquad a = \langle\alpha\rangle + m + d + 2. \tag{1.82}$$

As $\rho_\alpha(1) = 0$ it follows that τ is homogeneous of degree m if, and only if, we have

$$\int e^{as} h'(s) ds = 0 \qquad \text{for } a = m + d + 2, \ldots, m + d + 2 + k. \tag{1.83}$$

Suppose now that $m \notin \mathbb{Z}$. Then (1.83) is satisfied by

$$h'(s) = \prod_{a=m+d+2}^{m+d+2+k} (\frac{1}{a} \frac{d}{dt} + 1) g(s), \tag{1.84}$$

with $g \in C_c^\infty(\mathbb{R}^{d+1})$ such that $\int g(t) dt = 1$. So τ defined by (1.79) with $\psi(\mu) = \int_{\log \mu}^\infty h'(s) ds$ is a homogeneous extension of f. Moreover, if $\tau_1 \in \mathcal{S}'(\mathbb{R}^{d+1})$ is another homogeneous extension of f then $\tau - \tau_1$ is supported at 0 and we have $\tau = \tau_1 + \sum a_\alpha \delta^{(\alpha)}$, $a_\alpha \in \mathbb{C}$. As both τ and τ_1 are homogeneous, it follows

$$\sum (\lambda^{-(d+2-\langle\alpha\rangle)} - \lambda^m) a_\alpha \delta^{(\alpha)} = 0. \tag{1.85}$$

Thus $a_\alpha = 0$ for any α and $\tau_1 = \tau$.

It remains to treat the case m integer $\leq -(d+2)$. In this case we can set $k = -(m + d + 2)$ and take h of the form

$$h'(s) = \prod_{a=m+d+2}^{1} (\frac{1}{a} \frac{d}{dt} + 1) g(s), \tag{1.86}$$

with $g \in C_c^\infty(\mathbb{R}^{d+1})$ such that $\int g(t)dt = 1$. Then $\rho_\alpha(\lambda) = 0$ for $\langle\alpha\rangle < -(m+d+2)$. For $\langle\alpha\rangle = -(m = d+2)$ we have

$$\frac{d}{ds}\rho_\alpha(e^s) = -\int h'(s)ds = 1. \tag{1.87}$$

Thus $\rho_\alpha(\lambda) = \log\lambda$ and we have

$$\tau(\lambda.\xi) - \lambda^m\tau(\xi) = \lambda^m \log\lambda \sum_{\langle\alpha\rangle = -(m+d+2)} c_\alpha(f)\delta^{(\alpha)}(\xi), \qquad \lambda > 0. \tag{1.88}$$

Hence τ is homogeneous if $c_\alpha(f) = 0$ for $\langle\alpha\rangle = -(m = d+2)$.

Conversely, suppose there exists $\tau_1 \in \mathcal{S}'(\mathbb{R}^{d+1})$ homogeneous of degree m agreeing with f on $\mathbb{R}^{d+1}\setminus 0$. Then $\tau - \tau_1$ is supported at 0 and $\tau = \tau_1 + \sum a_\alpha \delta^{(\alpha)}$, $a_\alpha \in \mathbb{C}$. Thus

$$\tau(\lambda.\xi) - \lambda^m\tau(\xi) = \sum(\lambda^{-(d+2-\langle\alpha\rangle)} - \lambda^m)a_\alpha\delta^{(\alpha)} = \lambda^m\log\lambda \sum_{\langle\alpha\rangle=-(m+d+2)} c_\alpha(f)\delta^{(\alpha)}(\xi), \tag{1.89}$$

which implies that $c_\alpha(f) = 0$ for $\langle\alpha\rangle = -(m+d+2)$. ∎

Remark 1.24 If we take the inverse Fourier transform of the distribution τ constructed above we obtain a tempered distribution smooth away from zero. If m is not an integer $\leq -(m+d+2)$ it is homogeneous of degree $\widehat{m} = -(m+d+2)$. If m is an integer $\leq -(m+d+2)$ we have

$$(\check{\tau})(\lambda.x) = \lambda^{\widehat{m}}\check{\tau}(x) - (2\pi)^{-(d+2)}\lambda^{\widehat{m}}\log\lambda \sum_{\langle\alpha\rangle=\widehat{m}} c_\alpha(f)x^\alpha, \qquad \lambda > 0. \tag{1.90}$$

This leads to define the following spaces of distributions:

Definition 1.25 $\mathcal{K}_m(U\times\mathbb{R}^{d+1})$, $m \in \mathbb{C}$, *is the space of distributions* $K(x,y) \in C^\infty(U)\hat{\otimes}\mathcal{S}'(\mathbb{R}^{d+1})$ *which are smooth on* $U \times (\mathbb{R}^{d+1}\setminus 0)$ *and such that for any* $\lambda > 0$ *we have*

$$K(x,\lambda.y) = \lambda^m K(x,y), \qquad \text{if } m \notin \mathbb{N}, \tag{1.91}$$

$$K(x,\lambda.y) = \lambda^m K(x,y) + \lambda^m\log\lambda \sum_{\langle\alpha\rangle=m} c_\alpha(x)y^\alpha, \qquad \text{if } m \in \mathbb{N}. \tag{1.92}$$

Remark 1.26 If m is not a positive integer the restriction of K to $U \times (\mathbb{R}^{d+1}\setminus 0)$ is smooth and homogeneous of degree m. If m is a positive integer the equality (1.92) for $\lambda = \|y\|^{-1}$, $y \neq 0$, gives

$$K(x,y) = \|y\|^m K(x,\|y\|^{-1}.y) - \sum c_\alpha(x)y^\alpha\log\|y\|, \qquad y \neq 0. \tag{1.93}$$

Definition 1.27 *$\mathcal{K}^m(U \times \mathbb{R}^{d+1})$, $m \in \mathbb{C}$, is the space of distributions $K \in \mathcal{D}'(U \times \mathbb{R}^{d+1})$ with an asymptotic expansion*

$$K(x,y) \sim \sum_{j \geq 0} K_{m+j}(x,y), \qquad K_l \in \mathcal{K}_l(U \times \mathbb{R}^{d+1}), \tag{1.94}$$

in the sense that for integer N if J is sufficiently large we have

$$K - \sum_{j \leq J} K_{m+j} \in C^N(U \times \mathbb{R}^{d+1}). \tag{1.95}$$

Combining the definition with remark 1.26 we obtain:

Proposition 1.28 *Let $K \in \mathcal{K}^m(U \times \mathbb{R}^{d+1})$. Then:*

1) The distribution K lies in $C^\infty(U)\hat{\otimes}\mathcal{D}'(\mathbb{R}^{d+1})$ and it is smooth on $U \times (\mathbb{R}^{d+1} \backslash 0)$.

2) Near $y = 0$ we have

$$K(x,y) = \sum_{0 \leq j \leq -\Re m} a_{m+j}(x,y) - c_K(x) \log \|y\| + \mathrm{O}(1), \qquad \text{if } m \in \mathbb{N}, \tag{1.96}$$

$$K(x,y) = \sum_{0 \leq j \leq -\Re m} a_{m+j}(x,y) + \mathrm{O}(1), \qquad \text{if } m \notin \mathbb{N}. \tag{1.97}$$

In both cases $a_k(x,y)$ is a smooth function on $U \times (\mathbb{R}^{d+1} \backslash 0)$ homogeneous of degree k in the variable y.

Proposition 1.29 ([BG]) *Let P be a continuous operator from $C_c^\infty(U)$ into $C^\infty(U)$. Then P is a $\Psi_{\mathcal{V}}DO$ of order m if, and only if, its kernel is of the form*

$$k_P(x,y) = |\varepsilon_x'| K(x, -\varepsilon_x(y)) + R(x,y) \tag{1.98}$$

with $K \in \mathcal{K}^{\hat{m}}(U \times \mathbb{R}^{d+1})$, $\hat{m} = -(m+d+2)$, and $R \in C^\infty(U \times U)$.

Proof. Let $f \in S^*(U \times \mathbb{R}^{d+1})$. By definition the kernel of the operator $f(x, \sigma(x,D))$ is

$$k(x,y) = (2\pi)^{-(d+1)} \int e^{i\langle x-y, \xi\rangle} f(x, \sigma(x,\xi)) d\xi. \tag{1.99}$$

As $\sigma(x,\xi) = (\varepsilon_x'^t)^{-1}\xi$ we have

$$\int e^{i\langle x,\xi\rangle} f(x,\sigma(x,\xi))d\xi = |\varepsilon_x'| \int e^{i\langle \varepsilon_x'(x-y),\xi\rangle} f(x,\xi)d\xi = (2\pi)^{-(d+1)} \check{f}_{\xi \to y}(x, -\varepsilon_x(y)). \tag{1.100}$$

Therefore the proof reduces to show that a distribution K on $U \times \mathbb{R}^{d+1}$ lies in $\mathcal{K}^{\hat{m}}(U \times \mathbb{R}^{d+1})$ if, and only if, it is of the form

$$K(x, y) = \check{f}_{\xi \to y}(x, y) + R(x, y), \tag{1.101}$$

with $f \in S^m(U \times \mathbb{R}^{d+1})$ and $R \in C^\infty(U \times \mathbb{R}^{d+1})$.

However if $f \in S_m(U \times \mathbb{R}^{d+1})$ then by lemma 1.23 we can extend it into a distribution $\tau \in C^\infty(U)\hat{\otimes}\mathcal{S}'(\mathbb{R}^{d+1})$ such that $\check{\tau}_{\xi \to y}$ lies in $\mathcal{K}_{\hat{m}}(U \times \mathbb{R}^{d+1})$. Conversely if the distribution K lies in $\mathcal{K}_{\hat{m}}(U \times \mathbb{R}^{d+1})$ then $\tau = \hat{K}_{y \to \xi}$ is smooth outside on $U \times \mathbb{R}^{d+1}$ and satisfies

$$\tau(x, \lambda.\xi) = \lambda^m \tau(x, \xi) + \lambda^m \log \lambda \sum_{\langle \alpha \rangle = \hat{m}} c_\alpha \delta^{(\alpha)}, \qquad \lambda > 0. \tag{1.102}$$

Hence the restriction of τ to $U \times (\mathbb{R}^{d+1} \backslash 0)$ lies in $S_m(U \times \mathbb{R}^{d+1})$.

As by remark 1.7 the Fourier transform induces an equivalence between the asymptotic expansion for symbols and the asymptotic expansion for kernels, it follows that the form (1.101) characterizes the elements of $\mathcal{K}^m(U \times \mathbb{R}^{d+1})$ among the distributions on $U \times \mathbb{R}^{d+1}$. ■

Corollary 1.30 *Let P be a $\Psi_{\mathcal{V}}DO$ of integral order m. Then near the diagonal its kernel $k_P(x, y)$ has the following behavior*

$$k_P(x, y) = \sum_{-(m+d+2) \leq j \leq 0} a_j(x, \varepsilon_x(y)) - c_P(x) \log \|\varepsilon_x(y)\| + \mathrm{O}(1), \tag{1.103}$$

with $a_j(x, z)$ homogeneous of degree j in the variable z and $c_P(x)$ given by

$$c_P(x) = \frac{|\varepsilon'_x|}{(2\pi)^{(d+2)}} \int_{\|\xi\|=1} f_{-(d+2)}(x, \xi) i_E d\xi, \tag{1.104}$$

where $f_{-(d+2)}$ is the homogeneous symbol of degree $-(d+2)$ of P.

We can also define almost homogeneous distributions in the following way.

Definition 1.31 *$\mathcal{K}^m_{ah}(U \times \mathbb{R}^{d+1})$, $m \in \mathbb{C}$, is the space of distributions $K \in C^\infty(U)\hat{\otimes}\mathcal{D}'(\mathbb{R}^{d+1})$ which are smooth on $U \times (\mathbb{R}^{d+1} \backslash 0)$ and almost homogeneous in the second variable, i.e.*

$$K(x, t.y) - t^m K(x, y) \in C^\infty(U \times \mathbb{R}^{d+1}), \qquad t > 0. \tag{1.105}$$

Lemma 1.32 *Let $m \in \mathbb{C}$. Then we have*

$$\mathcal{K}^m_{ah}(U \times \mathbb{R}^{d+1}) = \mathcal{K}_m(U \times \mathbb{R}^{d+1}) + C^\infty(U \times \mathbb{R}^{d+1}). \tag{1.106}$$

Proof. As $\mathcal{K}_m(U \times \mathbb{R}^{d+1}) \subset \mathcal{K}_{ah}(U \times \mathbb{R}^{d+1})$ we need only to prove the inclusion

$$\mathcal{K}_{ah}^m(U \times \mathbb{R}^{d+1}) \subset \mathcal{K}_m(U \times \mathbb{R}^{d+1}) + C^\infty(U \times \mathbb{R}^{d+1}). \tag{1.107}$$

Let $K \in \mathcal{K}_{ah}(U \times \mathbb{R}^{d+1})$ and $\varphi \in C_c^\infty(\mathbb{R}^{d+1})$ such that $\varphi = 1$ near zero. Then the distribution $\varphi(y)K(x,y)$ lies in $C^\infty(U)\hat{\otimes}\mathcal{E}'(\mathbb{R}^{d+1})$ and we define a smooth function on $U \times \mathbb{R}^{d+1}$ by setting

$$f(x,\xi) = (\varphi K)^\wedge_{y\to\xi}(x,\xi). \tag{1.108}$$

Moreover if we set $\hat{m} = -(m+d+2)$ then for any $t > 0$ the symbol $t^{d+2}(f(x,t.\xi) - t^{\hat{m}}f(x,\xi))$ lies in $S^{-\infty}(U \times \mathbb{R}^{d+1})$ for it is the Fourier transform in the second variable of

$$\begin{aligned} & \varphi(t.y)K(x,t.y) - t^m\varphi(y)K(x,y) \\ = \;& \varphi(t.y)(K(x,t.y) - t^m K(x,y)) + (\varphi(t.y) - \varphi(y))K(x,y). \end{aligned} \tag{1.109}$$

Thus f is almost homogeneous of degree $\hat{m}$ and by lemma 1.10 there exists $g \in S_{h,\hat{m}}(U \times \mathbb{R}^{d+1})$ such that $f - g$ has rapid decay at infinity. Then by lemma 1.23 we can extend g into a distribution $\tau \in C^\infty(U)\hat{\otimes}\mathcal{S}'(\mathbb{R}^{d+1})$ such that $\check{\tau}_{\xi\to y}$ belongs to $\mathcal{K}_m(U \times \mathbb{R}^{d+1})$. As $\check{f}_{\xi\to y} - \check{\tau}_{\xi\to y}$ is smooth we can conclude that K coincides with an element of $\mathcal{K}_m(U \times \mathbb{R}^{d+1})$ up to a smooth function. ■

1.7 Invariance by Heisenberg diffeomorphisms

Proposition 1.33 ([BG]) *Let $\phi : U \to \tilde{U}$ be a Heisenberg diffeomorphism, where $\tilde{U}$ is another open subset of $\mathbb{R}^{d+1}$ equipped with a hyperplane bundle $\tilde{\mathcal{V}} \subset T\tilde{U}$ and a $\tilde{\mathcal{V}}$-frame of $T\tilde{U}$. Then for any $\tilde{P} \in \Phi^m_{\tilde{\mathcal{V}}}(\tilde{U})$ the pullback operator $P = \phi^*\tilde{P}$ is a $\Psi_{\mathcal{V}}DO$ of order m on U. Moreover,*

$$c_P(x) = |\phi'(x)| c_{\tilde{P}}(\phi(x)), \qquad x \in U, \tag{1.110}$$

where $c_P(x)$ and $c_{\tilde{P}}(\tilde{x})$ are the coefficients of the logarithmic divergencies of the kernels of P and $\tilde{P}$ given by corollary 1.30.

The proof requires the following analysis of the action of smooth functions on $\mathcal{K}^*(U \times \mathbb{R}^{d+1})$.

Lemma 1.34 ([BG]) *Let $K \in \mathcal{K}^m(U \times \mathbb{R}^{d+1})$ and $f \in C^\infty(U \times \mathbb{R}^{d+1})$. Then:*

1) The distribution $f(x,y)K(x,y)$ lies in $\mathcal{K}^m(U \times \mathbb{R}^{d+1})$.

2) If for some integer k we have $f(x,y) = \mathrm{O}(\|y\|^k)$ near $y = 0$, then fK is in $\mathcal{K}^{m+k}(U \times \mathbb{R}^{d+1})$.

Proof. By Taylor formula there exist smooth functions g_α such that
(1.111)
$$f(x,y)K(x,y) = \sum_{\langle\alpha\rangle<N} \frac{y^\alpha}{\alpha!}\partial_y^\alpha f^{(\alpha)}(x,0)K(x,y) + \sum_{\langle\alpha\rangle=N} \frac{y^\alpha}{\alpha!} g_\alpha(x,y)K(x,y).$$

The distribution $\frac{y^\alpha}{\alpha!}\partial_y^\alpha f^{(\alpha)}(x,0)K(x,y)$ lies in $\mathcal{K}^{m+\langle\alpha\rangle}(U \times \mathbb{R}^{d+1})$ and the remainder term

$$R_N(x,y) = \sum \frac{1}{\alpha!} y^\alpha g_\alpha K(x,y), \tag{1.112}$$

is smooth outside $U \times 0$.

However by proposition 1.28 there exists a real μ such that for any multi-order β we have

$$|\partial_x^\beta K(x,y)| = \mathrm{O}(\|y\|^{-\mu}) \qquad \text{near } y = 0. \tag{1.113}$$

It follows that $R_N(x,y)$ is smoother and smoother as $N \to \infty$. Thus

$$f(x,y)K(x,y) \sim \sum \frac{y^\alpha}{\alpha!}\partial_y^\alpha f^{(\alpha)}(x,0)K(x,y), \tag{1.114}$$

which implies that fK lies in $\mathcal{K}^m(U \times \mathbb{R}^{d+1})$.

Finally if near $y = 0$ we have $f(x,y) = \mathrm{O}(\|y\|^k)$ or some integer k, then $\partial_y^\alpha f(x,0) = 0$ for $\langle\alpha\rangle < k$ and fK is actually in $\mathcal{K}^{m+k}(U \times \mathbb{R}^{d+1})$. ■

Proof of proposition 1.33. By proposition 1.29 the kernel of $\tilde{P}$ is of the form

$$k_{\tilde{P}}(\tilde{x},\tilde{y}) = |\tilde{\varepsilon'_x}|\tilde{K}(\tilde{x},\tilde{\varepsilon}_{\tilde{x}}(\tilde{y})) + \tilde{R}(\tilde{x},\tilde{y}), \tag{1.115}$$

with $\tilde{K} \in \mathcal{K}^{\hat{m}}(\tilde{U} \times \mathbb{R}^{d+1})$ and $\tilde{R} \in C^\infty(\tilde{U} \times \tilde{U})$. Here $\tilde{\varepsilon}_{\tilde{x}}$ is the $\tilde{x}$-coordinates map with respect to the $\tilde{\mathcal{V}}$-frame of $T\tilde{U}$. So P has kernel

$$k_P(x,y) = |\phi'(y)|k_{\tilde{P}}(\phi(x),\phi(y)) = |\varepsilon'_x|K(x,\varepsilon_x(y))| + R(x,y), \tag{1.116}$$

with $R \in C^\infty(U \times U)$ and K given by

$$K(x,y) = |\phi'_x(y)|\tilde{K}(\phi(x),\phi_x(y)), \qquad \phi_x = \tilde{\varepsilon}_{\phi(x)} \circ \phi \circ \varepsilon_x^{-1}. \tag{1.117}$$

Note that ϕ_x is the diffeomorphism ϕ expressed in the x-coordinates and $\tilde{x}$-coordinates for $\tilde{x} = \phi(x)$.

However as $\phi'(x)$ maps $\mathcal{V}_x$ into $\mathcal{V}_{\phi(x)}$ the tangent map $\phi'_x(0)$ should map $\operatorname{span}\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_1}\}$ into itself. Thus $\phi'_x(0)$ is necessarily of form

$$\phi'_x(0) = \begin{pmatrix} a_{00} & 0 & \cdots & 0 \\ a_{10} & & & \\ \vdots & & A_{||} & \\ a_{d0} & & & \end{pmatrix}, \tag{1.118}$$

with $a_{00} \neq 0$ and $A_{||} \in GL_d(\mathbb{R})$. It follows that we can write ϕ_x as

$$\phi_x(y) = \widehat{\phi}_x(y) + \theta_x(y), \tag{1.119}$$

where $\widehat{\phi}_x$ and θ_x depends smoothly on x, $\widehat{\phi}_x$ is polynomial and homogeneous in y in the sense that

$$\widehat{\ }\phi'_x(0)(\lambda.y) = \lambda.\phi'_x(0)y, \qquad \lambda > 0, \tag{1.120}$$

and θ_x is such that near $y = 0$ we have

$$\theta_x(y) = (\mathrm{O}(|y_0|^2 + |y_0||y'| + |y'|^3), \mathrm{O}(|y_0|^2 + y'|^2)) = (\mathrm{O}(\|y\|^3), \mathrm{O}(\|y\|^2)). \tag{1.121}$$

Then pick a smooth family $(\chi_x)_{x\in U} \subset C_c^\infty(\mathbb{R}^{d+1})$ such that

$$\operatorname{supp}\chi_x \subset U, \qquad \|\chi_x(\varepsilon_x(y))\theta_x(y)\| < \|\widehat{\phi}_x(y)\|. \tag{1.122}$$

As $\widehat{\phi}_x + \chi_x(\varepsilon_x(y))\theta_x(y)$ vanishes nowhere and coincides with ϕ_x near $y = 0$, we have

$$K(x, y) = |\psi'_x(y)|\tilde{K}(\phi(x), \widehat{\phi}_x + \chi_x(\varepsilon_x(y))\theta_x(y)) \qquad \bmod C^\infty. \tag{1.123}$$

So we may suppose that ϕ_x and θ_x are defined on the whole $\mathbb{R}^{d+1}$ and we have $\|\theta_x(y)\| < \|\widehat{\phi}_x\|$. Then by proposition 1.29 it is enough to check that K lies in $\mathcal{K}^{\hat{m}}(U \times \mathbb{R}^{d+1})$.

Now the Taylor expansion for $K(x, y)$ gives

$$K(x, y) = \sum_{\langle\alpha\rangle < N} |\psi'_x(y)| \frac{\theta_x(y)^\alpha}{\alpha!} \partial_y^\alpha \tilde{K}(\phi(x), \widehat{\phi}_x) + R_N(x, y), \tag{1.124}$$

with the remainder term given by

$$R_N(x, y) = \sum_{\langle\alpha\rangle = N} |\psi'_x(y)| \frac{\theta_x(y)^\alpha}{\alpha!} \int_0^1 \partial_y^\alpha \tilde{K}(\phi(x), \widehat{\phi}_x + t\theta_x(y))dt. \tag{1.125}$$

As $|\theta_x(y)^\alpha| = \mathrm{O}(\|y\|^{3\alpha_0 + 2|\alpha'|}) = \mathrm{O}(\|y\|^{\frac{3}{2}\langle\alpha\rangle})$ the lemma 1.34 implies that

$$|\psi'_x(y)|\theta_x(y)^\alpha \partial_y^\alpha \tilde{K}(\phi(x), \widehat{\phi}_x) \in \mathcal{K}^{\hat{m} + \frac{\langle\alpha\rangle}{2}}(U \times \mathbb{R}^{d+1}). \tag{1.126}$$

Moreover R_N is smooth on $U \times (\mathbb{R}^{d+1} \backslash 0)$ and by proposition 1.28 for N large enough we have

$$|R_N(x,y)| = \mathrm{O}(\|y\|^{\frac{3N}{2}} \|y\|^{\Re \hat{m} - N}) = \mathrm{O}(\|y\|^{\Re \hat{m} + \frac{3N}{2}}). \tag{1.127}$$

Hence R_N is smoother and smoother as $N \to \infty$. We can then conclude that K lies in $\mathcal{K}^{\hat{m}}$ and P is a $\Psi_{\mathcal{V}}DO$ on U of order $\hat{m}$.

Finally working out the Taylor expansions (1.111) and (1.124) we obtain

$$K(x,y) = |\phi_x'(0)| \tilde{K}(\phi(x), \widehat{\phi}_x(y)) + \sum y_j K^{(j)}(x,y), \qquad K^{(j)} \in \mathcal{K}^*(U \times \mathbb{R}^{d+1}). \tag{1.128}$$

As $y_j K^{(j)}(x,y)$ cannot have a logarithmic divergency, the logarithmic divergency of K is

$$-|\phi_x'(0)| c_{\tilde{K}}(\phi(x)) \log \|\widehat{\phi}_x\| = -|\phi_x'(0)| c_{\tilde{K}}(\phi(x)) \log \|y\| + 0(1). \tag{1.129}$$

Hence $c_P(x) = |\phi(x)| c_{\tilde{P}}(\phi(x))$ and the proof is complete. ■

1.8 Ellipticity and parametrices

Definition 1.35 *Let $P \in \Psi_{\mathcal{V}}^m(U)$ with principal symbol $f_m \in S_m(U \times \mathbb{R}^{d+1})$. Then P is said to be elliptic in the Heisenberg calculus if there exists some $g_{-m} \in S_{-m}(U \times \mathbb{R}^{d+1})$ such that*

$$f_m * g_{-m} = 1 = g_{-m} * f_m. \tag{1.130}$$

Proposition 1.36 *Let $P \in \Psi_{\mathcal{V}}^m(U)$. Then:*

1) The $\Psi_{\mathcal{V}}DO$ operator P is elliptic if, and only if, there exists $Q \in \Psi_{\mathcal{V}}^{-m}(U)$ such that

$$PQ = 1 = QP \qquad \bmod \Psi^{-\infty}(U). \tag{1.131}$$

2) If P is elliptic in the Heisenberg calculus then it is a hypoelliptic operator, i.e. for any $u \in \mathcal{E}'(U)$ we have

$$Pu \text{ smooth near } x_0 \implies u \text{ smooth near } x_0. \tag{1.132}$$

Remark 1.37 Set $k = \frac{1}{2}\Re m$ if $\Re m \geq 0$ and $k = \Re m$ otherwise. If $P \in \Psi_{\mathcal{V}}^m(U)$ is elliptic it follows from proposition 1.13 that for any $u \in \mathcal{E}(U)'$ we have $Pu \in H^s_{\mathrm{loc}}(U) \Rightarrow\ u \in H^{s+k}_{\mathrm{loc}}(U)$.

Let us now give an ellipticity condition for a sublaplacian Δ on U in the form

$$\Delta = -\sum_{j=1}^{d} X_j^2 - i\lambda(x)X_0 + \sum_{j=1}^{d} \mu_j(x)X_j + \nu(x), \tag{1.133}$$

where λ, $\mu_1, \ldots, \mu_d$ and ν are smooth functions. Let $y \in U$. In the y-coordinates the vector fields X_j take the form

$$X_j = \frac{\partial}{\partial x^j} + \frac{1}{2}\sum_{l=0}^{d} \beta_{jl}(x)\frac{\partial}{\partial x^k}, \ldots j = 1, \ldots, d, \tag{1.134}$$

with $\beta_{jl}(0) = 0$. Then the y-invariant vector fields are given by

$$X_0^y = \frac{\partial}{\partial x^0}, \qquad X_j^y = \frac{\partial}{\partial x^j} + \frac{1}{2}\sum_{k=1}^{d} b_{jk}x^k \frac{\partial}{\partial x^0}, \qquad j = 1, \ldots, d, \tag{1.135}$$

where $b_{jk} = \frac{\partial}{\partial x^k}\beta_{j0}(0)$.

Consider now the 1-form θ annihilating $\mathcal{V}$ such that $\theta(X_0) = 1$ and let L be the 2-form

$$L = -d\theta(X, Y) = \theta([X, Y]) \qquad X, Y \in \mathcal{V}. \tag{1.136}$$

Thus for any X, Y in $\mathcal{V}$ we have

$$[X, Y] = L(X, Y)X_0 \qquad \mod \mathcal{V}. \tag{1.137}$$

In particular at y we have $L_y(X_j, X_k) = b_{jk} - b_{kj}$.

Let $a_1(y), \ldots, a_d(y)$ be the eigenvalues of the skew-symmetric matrix with entries $c_{jk} = b_{jk} - b_{kj}$ listed so that $a_j(y) > 0$ and $a_{n+j}(y) = -a_j(y)$ for $j = 1, \ldots n$, and $a_{2n+k}(y) = 0$ for $k = 1, \ldots, d - 2n$. Then the *singular set* $\Lambda^y \subset \mathbb{C}$ is defined by

$$\Lambda^y = \{\lambda \in \mathbb{R}\,;\ |\lambda| \geq \sum a_j(y)\} \qquad \text{if } 2n < d, \tag{1.138}$$

$$\Lambda^y = \{\pm \sum (2\alpha_j + 1)a_j(y)\,;\ \alpha \in \mathbb{N}^d\} \qquad \text{if } 2n = d. \tag{1.139}$$

Note that by (1.137) the definition of Λ^y does not depend on the choice of the $\mathcal{V}$-frame.

Proposition 1.38 ([BG]) *Let Δ be the sublaplacian (1.133) and define the singular sets Λ^y as above. Then the following conditions are equivalent:*

(i) For any $y \in U$ the coefficient $\lambda(y)$ does not lie in the singular set Λ^y.

(ii) For any $y \in U$ the left-invariant operator Δ_2^y is invertible.

(iii) Δ is elliptic in the Heisenberg calculus.

Furthermore if one these conditions holds the principal symbol of a parametrix for Δ is

$$f_{-2}(y,\xi) = f_{-2}^y(\xi), \tag{1.140}$$

where $f_{-2}^y(\xi)$ is the symbol of the inverse of Δ_2^y.

Remark 1.39 Since the singular sets are contained in $\mathbb{R}$, the above theorem implies in the Heisenberg calculus setting the Hörmander theorem [Ho2]. The latter asserts that given real vector fields $X_0, X_1, \ldots, X_m$ generating TM together with their brackets the differential operator $\Delta = -\sum_{j=1}^m X_j^2 + X_0$ is hypoelliptic.

Remark 1.40 In the rest of the dissertation we will used only the weaker ellipticity condition:

$$|\Re\lambda(y)| < \frac{1}{2}\sum_{j=1}^d |a_j(y)|, \qquad y \in U, \tag{1.141}$$

and for technical reasons we shall refer to it as being precisely the ellipticity condition for Δ given by the proposition 1.38.

1.9 The $\Psi_{\mathcal{V}}DO$ operators on manifolds

Let $(M,\mathcal{V})$ be a Heisenberg manifold and $\mathcal{E}$ be a vector bundle over M. Then proposition 1.33 allows us to define $\Psi_{\mathcal{V}}DO$ operators on M acting on the sections of $\mathcal{E}$.

Definition 1.41 *$\Psi_{\mathcal{V}}^m(M)$, $m \in \mathbb{C}$, is the space of continuous operators P from $C_c^\infty(M)$ into $C^\infty(M)$ such that:*

(i) the distribution kernel of P is smooth outside the diagonal of $M \times M$;

(ii) on any Heisenberg chart P is given by a $\Psi_{\mathcal{V}}DO$ of order m on an open subset of $\mathbb{R}^{d+1}$ equipped with a $\mathcal{V}$-frame.

Definition 1.42 *$\Psi_{\mathcal{V}}^m(M,\mathcal{E})$, $m \in \mathbb{C}$, is the space of continuous operators P from $C_c^\infty(M,\mathcal{E})$ into $C^\infty(M,\mathcal{E})$ such that:*

(i) the distribution kernel of P is smooth outside the diagonal of $M \times M$.

(ii) on any trivializing Heisenberg chart P is given by a matrix of $\Psi_{\mathcal{V}}DO$ operators of order m on an open subset of $\mathbb{R}^{d+1}$ equipped with a $\mathcal{V}$-frame.

All the preceding results in the case an open subset of $\mathbb{R}^{d+1}$ hold for $\Psi_{\mathcal{V}}DO$'s on M. Moreover it follows from proposition 1.33 that the coefficient $c_P(x)$ of the logarithmic divergency of $\Psi_{\mathcal{V}}DO$ with integral order can be globally defined as a density.

Proposition 1.43 *Let $P \in \Psi_{\mathcal{V}}^m(M, \mathcal{E})$, $m \in \mathbb{Z}$. Then:*

1) On a trivializing Heisenberg chart the kernel $k_P(x, y)$ of P has the following behavior near the diagonal

$$k_P(x, y) = \sum_{-(m+d+2)}^{0} a_j(x, \varepsilon_x(y)) - c_P(x) \log \|\varepsilon_x(y)\| + \mathrm{O}(1), \tag{1.142}$$

where ε_x is the x-coordinates map related to the chart, $a_j(x, z)$ is homogeneous of degree j in the variable z and $c_P(x)$ is a globally defined density on M with values in $\mathrm{END}\,\mathcal{E}$.

2) Let $\phi : (M, \mathcal{V}) \to (\tilde{M}, \tilde{\mathcal{V}})$ be a Heisenberg diffeomorphism. Then we have the equality

$$c_{\phi_* P}(\tilde{x}) = \phi_*(c_P)(\tilde{x}). \tag{1.143}$$

Proposition 1.44 ([BG]) *Suppose M is compact. Then:*

1) The class $\Psi_{\mathcal{V}}^*(M, \mathcal{E})$ is stable under the composition of operators.

2) Any $P \in \Psi_{\mathcal{V}}^m(M, \mathcal{E})$ with $\Re m \leq 0$ extends to a continuous endomorphism of $L^2(M, \mathcal{E})$. If furthermore $\Re m < 0$ this endomorphism is compact.

3) Any $P \in \Psi_{\mathcal{V}}^m(M, \mathcal{E})$, elliptic with $\Re m \geq 0$, is Fredholm and has its kernel contained in $C^\infty(M, \mathcal{E})$.

For our examples of sublaplacians on a pseudohermitian manifold we obtain:

Proposition 1.45 ([BG]) *Let (M^{2n+1}, θ) be a pseudohermitian manifold and let $\mathcal{V} = \ker \theta$. The following operators are elliptic in the $\Psi_{\mathcal{V}}DO$ calculus:*

(i) the Kohn Laplacian $\Box_b$ acting on (p, q)-forms with $0 < q < n$;

(ii) the pseudohermitian sublaplacian Δ_b;

(iii) the conformal sublaplacian $\boxdot_\theta$.

1.10 Heat kernel of an elliptic sublaplacian

Let $(M^{d+1}, \mathcal{V})$ be a compact Heisenberg manifold equipped with a smooth non-negative density and let Δ be an elliptic sublaplacian on M bounded from below. We can then define the heat semi-group $e^{-t\Delta}$, $t > 0$. It is defined on $L^2(M)$, strongly differentiable and for $t > 0$ the operator $e^{-t\Delta}$ maps to the domain of Δ,

$$\frac{d}{dt}e^{-t\Delta} = -\Delta e^{-t\Delta}; \qquad e^{-t\Delta}u \longrightarrow_{t\to 0^+} u, \qquad u \in L^2(M). \tag{1.144}$$

The point here is that the heat operator $e^{-t\Delta}$ provides an inverse for the differential operator $\frac{\partial}{\partial t} + \Delta$ on $M \times \mathbb{R}$. Indeed through the isomorphism $C_c^\infty(M \times \mathbb{R}) \simeq C_c^\infty(\mathbb{R}, C^\infty(M))$ the inverse is given by

$$Qu(t) = \int_{-\infty}^{t} e^{-(t-s)\Delta}u(s)ds, \qquad u \in C_c^\infty(\mathbb{R}, C^\infty(M)). \tag{1.145}$$

Therefore one could derive an asymptotic expansion for the kernel $k_t(x, y)$ of $e^{-t\Delta}$ by constructing a pseudodifferential inverse for $\frac{\partial}{\partial t}+\Delta$. This is precisely what is done in [BGS] and what we shall present in this section.

Let $\mathbb{C}_-$ be the half-plane $\{\operatorname{im}\tau < 0\}$ with closure $\bar{\mathbb{C}}_- = \{\operatorname{im}\tau \leq 0\}$. The relevant class of symbols for studying $\Delta + \partial_t$ is associate to dilations on $\mathbb{R}^{d+2} = \mathbb{R}^{d+1} \times \mathbb{R}$ defined by

$$\lambda.(x, t) = (\lambda.x, \lambda^2 t) = (\lambda^2.x_0, \lambda x_1, \ldots, \lambda x_d, \lambda^2 t), \qquad \lambda > 0. \tag{1.146}$$

Definition 1.46 *$S_{h,m}(U \times \mathbb{R}^{d+2})$, $m \in \mathbb{Z}$, consists in functions $f(x, \xi, \tau) \in C^\infty(U \times (\mathbb{R}^{d+2} \setminus 0))$ which extend to a function in $C^\infty(U \times (\mathbb{R}^{d+1} \times \bar{\mathbb{C}}_-) \setminus 0)$ in such way as to be holomorphic with respect to τ and to be homogeneous of degree m in the two last variables, i.e.*

$$f(x, \lambda.x, \lambda^2\tau) = \lambda^m f(x, \xi, \tau), \qquad \lambda > 0. \tag{1.147}$$

Definition 1.47 *$S_h^m(U \times \mathbb{R}^{d+2})$, $m \in \mathbb{Z}$, is the space of functions $f \in C^\infty(U \times \mathbb{R}^{d+2})$ which admit an asymptotic expansion of symbols $f \sim \sum f_{m-j}$ with $f_k \in S_{h,k}(U \times \mathbb{R}^{d+2})$.*

Definition 1.48 *For $m \in \mathbb{Z}$, $\Psi_{\mathcal{V},h}^m(U \times \mathbb{R})$ is the space of operators P from $C_c^\infty(U \times \mathbb{R})$ into $C^\infty(U \times \mathbb{R})$ of the form*

$$P = f(x, \sigma(x, D_x), D_t) + R, \tag{1.148}$$

with $f \in S_h^m(U \times \mathbb{R}^{d+2})$ and $R \in \Psi^{-\infty}(U \times \mathbb{R})$.

Now, all the preceding results concerning $\Psi_{\mathcal{V}}DO$ operators continue to hold in this context: convolution for homogeneous symbols, composition formula, kernel characterization, invariance by Heisenberg diffeomorphism and parametrix construction. In particular we can define such operators on any Heisenberg manifold.

However, there is an important specificity here due to the analyticity with respect to τ.

Lemma 1.49 ([BGS]) *Let $f(\xi,\tau) \in C^\infty(\mathbb{R}^{d+2} \setminus 0)$ be homogeneous of degree m, $m \in \mathbb{Z}$, which extends to an element of $C^\infty(\mathbb{R}^{d+1} \times \bar{\mathbb{C}}_- \setminus 0)$ in such way as to be holomorphic in the variable τ. Then f can be extended into a homogeneous distribution g on $\mathbb{R}^{d+2}$ such that $k(x,t) = \check{g}(x,t)$ vanishes for $t < 0$.*

As the converse follows from Paley-Wiener-Schwartz theorem we obtain a kernel characterization in terms of the following space of distributions:

Definition 1.50 *$\mathcal{K}_{h,m}(U \times \mathbb{R}^{d+2})$, $m \in \mathbb{Z}$, is the space of distributions $K \in C^\infty(U)\hat{\otimes}\mathcal{S}'(\mathbb{R}^{d+2})$ such that*

(i) $K(x,y,t)$ is smooth on $U \times (\mathbb{R}^{d+2} \setminus 0)$ and vanishes for $t < 0$.

(ii) $K(x, \lambda.y, \lambda^2 t) = \lambda^m K(x,y,t)$ for $\lambda > 0$.

Definition 1.51 *$\mathcal{K}_h^m(U \times \mathbb{R}^{d+2})$, $m \in \mathbb{Z}$, is the space of distributions $K \in \mathcal{D}'(U \times \mathbb{R}^{d+2})$ with an asymptotic expansion of kernels $K \sim \sum K_{m-j}$ with $K_l \in \mathcal{K}_l(U \times \mathbb{R}^{d+2})$.*

Using this calculus one can prove:

Proposition 1.52 ([BGS]) *Let $(M^{d+1}, \mathcal{V})$ be a compact Heisenberg manifold equipped with a smooth non-negative density and let Δ be an elliptic positive sublaplacian on M.*

1) The operator $\Delta + \frac{\partial}{\partial t}$ is invertible on $C_c^\infty(M \times \mathbb{R})$ and its inverse belongs to $\Psi_{\mathcal{V},h}^{-2}(U \times \mathbb{R})$.

2) Let $k(x,y,s-t)$ be the kernel of $(\Delta + \frac{\partial}{\partial t})^{-1}$. Then $k(x,y,t)$ is the kernel of $e^{-t\Delta}$ for $t > 0$ and $k(x,y,t) = 0$ for $t < 0$.

3) The operator $e^{-t\Delta}$ is smoothing for $t > 0$.

Proof. We refer to [BGS] for an explicit construction of a parametrix for $(\Delta + \frac{\partial}{\partial t})$ in $\Psi_{\mathcal{V},h}^{-2}(U \times \mathbb{R})$. We can anyway exhibit an inverse. Consider the operator on $C_c^\infty(M \times \mathbb{R}) \simeq C_c^\infty(\mathbb{R}, C^\infty(M))$ defined by

$$Qu(t) = \int_{-\infty}^{t} e^{-(t-s)\Delta} u(s) ds, \qquad u \in C_c^\infty(\mathbb{R}, C^\infty(M)). \tag{1.149}$$

By (1.144) we have

$$Q(\Delta + \frac{\partial}{\partial t}) = (\Delta + \frac{\partial}{\partial t})Q = 1. \tag{1.150}$$

Now let $Q_1 \in \Psi^{-2}_{\mathcal{V},h}(U \times \mathbb{R})$ be a parametrix for $\Delta + \frac{\partial}{\partial t}$, i.e.

$$(\Delta + \frac{\partial}{\partial t})Q_1 = 1 - R, \qquad R \in \Psi^{-\infty}(M \times \mathbb{R}). \tag{1.151}$$

Then we have

$$Q_1 = Q(\Delta + \frac{\partial}{\partial t})Q_1 = Q - QR. \tag{1.152}$$

As QR is smoothing we see that Q lies in $\Psi^{-2}_{\mathcal{V},h}(U \times \mathbb{R})$. Let $k(x, y, t - s)$ be the kernel of Q. Then

$$Qu(x,t) = \int_M \int_{\mathbb{R}} k_Q(x,y)u(y,s)ds, \qquad u \in C^\infty_c(M \times \mathbb{R}). \tag{1.153}$$

Identifying (1.149) and (1.153) we see that $k(x, y, t)$ is the kernel of $e^{-t\Delta}$ for $t > 0$ and $k(x, y, t) = 0$ for $t < 0$. As $k(x, y, t)$ is smooth on $M \times M \times (0, +\infty)$ it follows that $e^{-t\Delta}$ is smoothing for $t > 0$. ■

Proposition 1.53 ([BGS]) *Let $(M^{d+1}, \mathcal{V})$ be a compact Heisenberg manifold equipped with a smooth non-negative density and let Δ be an elliptic positive sublaplacian on M.*

1) *For t small the kernel of $e^{-t\Delta}$ has an asymptotic on the diagonal*

$$k_t(x,x) \sim t^{-\frac{d+2}{2}} \sum_{j \geq 0} a_j(x) t^j, \tag{1.154}$$

where the $a_j(x)$'s are smooth densities on M with $a_0(x)$ non-negative.

2) *For t small we have*

$$\operatorname{Trace}(e^{-t\Delta}) \sim t^{-\frac{d+2}{2}} \sum_{j \geq 0} t^j \int_M a_j(x). \tag{1.155}$$

3) *Let $\lambda_k(\Delta)$ be the k'th eigenvalue of Δ counted with multiplicity. Then for k large*

$$\lambda_k(\Delta) \sim (Ak)^{\frac{d+2}{2}}, \qquad A = \Gamma(1 + \frac{d+2}{2})^{-1} \int_M a_0(x). \tag{1.156}$$

Proof. By proposition 1.52 the operator $\Delta+\frac{\partial}{\partial t}$ is invertible on $C_c^\infty(M\times\mathbb{R})$ and its inverse $Q=(\Delta+\frac{\partial}{\partial t})^{-1}$ lies $\Psi_{\mathcal{V},h}^{-2}(U\times\mathbb{R})$. Moreover if we denote by $k_Q(x,y,s-t)$ the kernel of $(\Delta+\frac{\partial}{\partial t})^{-1}$ then $k_Q(x,y,t)$ is the kernel of $e^{-t\Delta}$ for $t>0$ and we have $k_Q(x,y,t)=0$ for $t<0$. Anyway, locally we have

$$k_Q(x,y,t)=K(x,-\varepsilon_x(y),t)+R(x,y,t), \tag{1.157}$$

with K a $\mathcal{K}_h{}^{-(d+2)}(U\times\mathbb{R}^{d+2})$-kernel, R smooth and U an open subset of $\mathbb{R}^{d+1}$ together with a $\mathcal{V}$-frame. Let $K(x,y)\sim\sum K_{-(d+2)+j}$, $K_l\in\mathcal{K}_{h,l}(U\times\mathbb{R}^{d+2})$, be the asymptotic expansion for K. Then if J is large enough

$$R_J(x,t)=k_Q(x,x,t)-|\varepsilon'_x|\sum_{j\leq J}K_{-(d+2)+j}(x,0,t)\in C^\infty(U)\hat{\otimes}C^N(\mathbb{R}). \tag{1.158}$$

As $R_J(x,t)=0$ for $t<0$ this implies

$$|R_J(x,t)|\leq C_{JN}t^N, \tag{1.159}$$

since we can take the constant uniform by shrinking U if necessary. Therefore letting

$$a_{\frac{j}{2}}(x)=|\varepsilon'_x|K_{-(d+2)+j}(x,0,1),\qquad j=0,1,\ldots, \tag{1.160}$$

we obtain

$$k_t(x,x)\sim t^{-\frac{(d+2)}{2}}\sum_{j\geq 0}t^{\frac{j}{2}}a_{\frac{j}{2}}(x). \tag{1.161}$$

This asymptotic holds on the whole M since we can globally define the $a_j(x)$'s as densities on M using the equalities

$$a_{\frac{j}{2}}(x)=\lim_{t\to 0^+}t^{\frac{(d+2)-j}{2}}(k_t(x,x)-t^{-\frac{(d+2)}{2}}\sum_{l<j}t^{\frac{l}{2}}a_{\frac{l}{2}}(x)). \tag{1.162}$$

Let us now show that $a_{\frac{j}{2}}(x)=0$ if j is odd. We work on U and we let $p_2(x,\xi,\tau)+i\tau$ be the principal symbol of $\Delta+\partial_t$. This symbol is invariant under the dilation by -1, i.e.

$$(-1).(\xi_0,\xi_1,\ldots,\xi_d,\tau)=(\xi_0,-\xi_1,\ldots,-\xi_d,\tau). \tag{1.163}$$

As the other symbols $p_1(x,\xi)$ and $p_0(x)$ don't depend on τ they are -1-homogeneous of degrees 1 and 0 respectively. On the other hand the principal symbol of $(\Delta+\partial_t)^{-1}$ is the solution of the equations

$$f_{-2}*(p_2+i\tau)=1=(p_2+i\tau)*f_{-2}. \tag{1.164}$$

So f_{-2} must be homogeneous of degree -2. By construction the symbol f_{-2-j} of $(\Delta+\partial_t)^{-1}$ with degree $-2-j$ is a homogeneous polynomial, with respect to the convolution $*$, in $f_{-2}(x,\xi,\tau)$ and the symbols of $\Delta+\partial_t$. So it has to be -1-homogeneous of degree $-2-j=j \mod 2$. As the jacobian of the dilation by -1 is equal to 1 we see that $K_{j-(d+2)}=\check{f}_{-2-j}$ is also -1-homogeneous of degree j. Hence $K_{j-(d+2)}(x,0,1)=(-1)^j K_{j-(d+2)}(x,0,1)$, which finally implies $a_{\frac{j}{2}}(x)=0$ for j odd.

Therefore for t small we have

$$k_t(x,x) \sim t^{-\frac{(d+2)}{2}} \sum_{j\geq 0} t^{\frac{j}{2}} a_{\frac{j}{2}}(x). \tag{1.165}$$

Integrating over M we obtain

$$\mathrm{Trace}(e^{-t\Delta}) \sim t^{-\frac{d+2}{2}} \sum_{j\geq 0} t^j \int_M a_j(x). \tag{1.166}$$

It remains to prove that $a_0(x)$ is non-negative. In [BGS, theorem 5.22] an explicit formula is given for $K_{-(d+2)}(x,y,t)$. For x fixed $K(x,.,.)$ is given up to a change of coordinates by the inverse Fourier transform of a symbol in the form

$$f^x_{-2}(\xi,\tau)=\int_0^\infty e^{i\tau s-\lambda\xi_0 s} G(\xi,s)ds, \qquad \lambda=\lambda(x), \tag{1.167}$$

where $G(\xi,s)$ is a non-negative function even in the variable s. Using the parity of G we get t

$$K_{-(d+2)}(x,0,1)=\frac{|\phi'_x|}{2}(2\pi)^{-(d+1)} \int e^{-\lambda\xi_0} G(\xi,1)d\xi>0, \tag{1.168}$$

where ϕ_x is the issued change of coordinates. It follows that a_0 is a non-negative density.

Finally, the asymptotic (1.166) together with the Tauberian theorem of Hardy-Littlewood show that for k large we have

$$\lambda_k(\Delta) \sim (Ak)^{\frac{d+2}{2}}, \qquad A=\Gamma(1+\frac{d+2}{2})^{-1}\int_M a_0(x), \tag{1.169}$$

which completes the proof. ■

Remark 1.54 We will show in chapter 2 that any selfadjoint elliptic sublaplacian on a compact Heisenberg manifold is bounded from below. So proposition 1.53 continue to hold in this case.

In the case of a pseudohermitian manifold we obtain:

Proposition 1.55 ([BGS]) *Let (M^{2n+1},θ) be a compact pseudohermitian manifold. Then proposition 1.53 hold for the following operators:*

(i) the Kohn Laplacian $\Box_b$ *acting on* (p,q)*-forms,* $0<q<n$,

(ii) the pseudohermitian sublaplacian Δ_b,

(iii) the conformal pseudohermitian sublaplacian $\boxdot_\theta$.

Moreover for each of these operators the coefficient $a_j(x)$ *in the asymptotic (1.154) takes the form*

$$a_j(x) = A_j(x)(d\theta)^n \wedge \theta, \qquad j \leq 0, \tag{1.170}$$

where $A_j(x)$ *is a universal polynomial in the jets of the components of the curvature and torsion forms of the Tanaka-Webster connection. In the case* $j=0$ *and* $j=1$ *we have*

$$A_0 = \alpha_n, \qquad A_1 = \beta_n R_n, \tag{1.171}$$

where α_n, β_n *are universal constants and* R_n *is the Tanaka-Webster scalar curvature.*

Chapter 2

Parametric $\Psi_{\mathcal{V}}DO$ operators and resolvent of an elliptic sublaplacian on a Heisenberg manifold

In this chapter we develop a suitable calculus for $\Psi_{\mathcal{V}}DO$'s with parameter which enables us to construct an asymptotic resolvent and to show the existence of rays of minimal growth for an elliptic sublaplacian. The situation is more complicated than for classical ΨDO operators. Roughly speaking the composition for homogeneous symbols in the Heisenberg calculus is not microlocal whereas it is just the pointwise product for functions for standard symbols of ΨDO operators.

To see this in more details let us first briefly recall the construction by Seeley [Se] of the complex powers of an elliptic operator with a ray of minimal growth, in the special case of the Laplacian Δ on a Riemannian manifold M. The powers Δ^s are defined by the integral

$$\Delta^s = \frac{i}{2\pi}\int_\Gamma \lambda^s(\Delta-\lambda)^{-1}d\lambda, \qquad \Re s<0, \tag{2.1}$$

where Γ is a curve starting at ∞, passing along the negative real axis to a small circle about the origin, then clockwise about the circle and back to ∞ along the negative real axis. If we suppose that 0 is not an eigenvalue for Δ then the negative real axis is a ray of minimal growth. This means that for any λ in this axis $\Delta-\lambda$ is invertible on $L^2(M)$ and the norm of the resolvent $\|(\Delta-\lambda)^{-1}\|$ is $\mathrm{O}(1/|\lambda|)$.

The idea for showing the above complex powers are ΨDO's is to introduce a suitable class of continuous operators on $C^\infty(M)$ parametrized by an open subset Λ of $\mathbb{C}$ containing the curve Γ. This class contains the resolvent $(\Delta-\lambda)^{-1}$ and possesses a symbolic calculus. The corresponding symbols are of the form

$$f_{(\lambda)}(x,\xi) = \sum_{j\geq 0} f_{(\lambda),m-j}(x,\xi), \qquad f_{(t^2\lambda),k}(x,t\xi) = t^k f_{(\lambda),k}(x,\xi,\lambda). \tag{2.2}$$

Whereas $f_{(\lambda)}$ is a family of smooth functions on $C^\infty(U\times\mathbb{R}^n)$ parametrized by Λ, the homogeneous symbols $f_{k(\lambda)}(x,\xi)$ cannot be defined on the whole set $U\times\mathbb{R}^n\times\Lambda$. For instance the principal symbol of $(\Delta-\lambda)^{-1}$ is $f_{(\lambda),-2}(x,\xi,\lambda) = (|\xi|^2-\lambda)^{-1}$ and is not defined for $|\lambda| = |\xi|^2$. The solution is then to define the homogenous symbols on

$$U\times\Theta = \{(x,\xi,\lambda)\in U\times\mathbb{R}^n\times\Lambda\ ;\ \Re\lambda<0 \text{ or } |\lambda|<|\xi|^2\}. \tag{2.3}$$

As the pointwise product of two functions defined on $U\times\Theta$ is still defined on $U\times\Theta$ we have a nice calculus for these symbols.

However, in the context of Heisenberg calculus the product of two homogenous symbols is obtained by means of oscillating integrals of the form

$$f_{1(\lambda)}*f_{2(\lambda)}(x,\xi) = (2\pi)^{-(d+1)}\iint e^{-i\langle z,\eta\rangle} f_{1(\lambda)}(x,\xi+\eta) f_{2(\lambda)}(x,\sigma^x(z,\xi))dzd\eta. \tag{2.4}$$

Thereby it is not obvious if $f_{1(\lambda)}$ and $f_{2(\lambda)}$ are defined on $U\times\Theta$ then so is $f_{1(\lambda)}*f_{2(\lambda)}$. This difficulty can be avoid if we consider instead almost homogenous symbols: they are defined on the whole $U\times\mathbb{R}^n\times\Lambda$ and $f_{(t^2\lambda),k}(x,t\xi) - t^k f_{(\lambda),k}(x,\xi)$ has rapid decay with respect to ξ.

The symbols and $\Psi_{\mathcal{V}}DO$ operators with parameter are introduced in section 2.1 and section 2.2. In section 2.3 the kernels of $\Psi_{\mathcal{V}}DO$'s with parameter are studied. This leads to a characterization which enables us to prove the invariance by Heisenberg diffeomorphism in section 2.4 and to define in section 2.5 $\Psi_{\mathcal{V}}DO$'s with parameter on any Heisenberg manifold . The section 2.6 is devoted to the construction of asymptotic resolvent for elliptic sublaplacians as a parametrix in the parametric $\Psi_{\mathcal{V}}DO$-calculus. This is used in section 2.7 to show the existence of rays of minimal growth for an elliptic sublaplacian (theorem 2.33). As corollary we obtain that any Heisenberg-elliptic selfadjoint sublaplacian on a compact Heisenberg manifold is actually bounded from below and thus has a heat kernel asymptotic as in [BGS] and proposition 1.53.

In all this chapter $\Lambda\subset\mathbb{C}\setminus 0$ is an open pseudocone (*cf.* definition 2.1 below) and U is an open subset of $\mathbb{R}^{d+1}$ together with a hyperplane bundle $\mathcal{V}\subset TU$ and a $\mathcal{V}$-frame $X_0,X_1,\ldots,X_d$. Then $\sigma(x,\xi)$ and ε_x denote the (real) symbol of the $\mathcal{V}$-frame and the affine change onto the x-coordinates.

2.1 Spaces of symbols with parameter

Definition 2.1 *1) A subset Λ of $\mathbb{C}\setminus 0$ is a pseudocone if for any $t\in(0,1)$ we have $t\Lambda\subset\Lambda$ and there exist a conical subset Θ and a bounded subset D such that $\Lambda=\Theta\cup D$.*

2) If Λ and Λ' are two pseudocones in $\mathbb{C}\backslash 0$ the notation $\Lambda' \subset\subset \Lambda$ means that up to the origin the closure of Λ' is contained in the interior of Λ.

The space of parameters is the following Fréchet space.

Definition 2.2 $\mathrm{Hol}^p(\Lambda)$*, $p \in \mathbb{Z}$, is the space of holomorphic functions $h : \Lambda \to \mathbb{C}$ such that for any pseudocone $\Lambda' \subset\subset \Lambda$ we have*

$$|h(\lambda)| \leq C_{\Lambda'}(1 + |\lambda|)^p, \qquad \lambda \in \Lambda'. \tag{2.5}$$

Its topology is defined by means of the seminorms given by the lower bounds on the constants in these estimates.

We define $\mathrm{Hol}^p(\Lambda)$-families with values in a topological vector space as follows.

Definition 2.3 *If E a locally convex topological vector space* $\mathrm{Hol}^p(\Lambda, E)$*, $p \in \mathbb{Z}$, is the space of* $\mathrm{Hol}^p(\Lambda)$*-families with values in E, i.e. holomorphic maps $h : \Lambda \to E$ such that, for any continuous seminorm q on E and any pseudocone $\Lambda' \subset\subset \Lambda$, we have*

$$|q(h(\lambda))| \leq C_{q\Lambda'}(1 + |\lambda|)^p, \qquad \lambda \in \Lambda'. \tag{2.6}$$

If $E = S^k_{||}(U \times \mathbb{R}^{d+1})$ or $E = S^{-\infty}(U \times \mathbb{R}^{d+1})$ we use instead the notations $S^{p,k}_{||}(U \times \mathbb{R}^{d+1}, \Lambda)$ and $S^{p,-\infty}(U \times \mathbb{R}^{d+1}, \Lambda)$.

Definition 2.4 *$S^p_m(U \times \mathbb{R}^{d+1}, \Lambda)$, $m, p \in \mathbb{Z}$, consists in* $\mathrm{Hol}^p(\Lambda)$*-families $f_{(\lambda)}$ of smooth functions on $U \times \mathbb{R}^{d+1}$ such that*

$$f_{(t^2\lambda)}(x, t.\xi) - t^m f_{(\lambda)}(x, \xi) \in S^{p,-\infty}(U \times \mathbb{R}^{d+1}, \Lambda), \qquad 0 < t < 1. \tag{2.7}$$

Remark 2.5 Interchanging the role of λ and $t^2\lambda$ we see that $f_{(t^2\lambda)}(x, t.\xi) - t^m f_{(\lambda)}(x, \xi)$ lies in $S^{-\infty}(U \times \mathbb{R}^{d+1})$ whenever $t^2\lambda \in \Lambda$. In particular if Θ is any open cone contained in Λ the family $f_{(t^2\lambda)}(x, t.\xi) - t^m f_{(\lambda)}(x, \xi)$ lies in $S^{p,-\infty}(U \times \mathbb{R}^{d+1}, \Theta)$ for any $t > 0$.

Lemma 2.6 *Let $m, p \in \mathbb{Z}$ and set $q = 2\max(0, -p)$. Then we have the inclusion*

$$S^p_m(U \times \mathbb{R}^{d+1}, \Lambda) \subset S^{p,m+q}_{||}(U \times \mathbb{R}^{d+1}, \Lambda). \tag{2.8}$$

Proof. We need to prove that for any pseudo-cone $\Lambda' \subset\subset \Lambda$ we have

$$|\partial^\alpha_x \partial^\beta_\xi f_{(\lambda)}(x, \xi)| \leq C_{\Lambda'\alpha\beta}(x)(1 + |\lambda|)^p(1 + \|\xi\|)^{m+q-\langle\beta\rangle}, \qquad \lambda \in \Lambda', \tag{2.9}$$

with $C_{\Lambda'\alpha\beta}(x)$ locally bounded on U. Set $\Lambda = \Theta \cup D$ with Θ conic and D bounded. We shall prove the estimates (2.9) separately for $\lambda \in \Theta$ and for $\lambda \in D$.

As Θ is conic for any $t > 0$ we have

$$f_{(t^2\lambda)}(x, t.\xi) - t^m f_{(\lambda)}(x,\xi) \in S^{p,-\infty}(U \times \mathbb{R}^{d+1}, \Theta) \tag{2.10}$$

By the arguments in the proof of lemma 1.10 there exists some $g_{(\lambda)}(x,\xi) \in \mathrm{Hol}^p(\Theta, C^\infty(\mathbb{R}^{d+1}\backslash 0))$ such that:

(i) $g_{(\lambda)}$ is homogeneous of degree m, i.e.

$$g_{(t^2\lambda)}(x, t.\xi) = t^m g_{(\lambda)}(x,\xi), \qquad \xi \neq 0, \quad \lambda \in \Theta, \tag{2.11}$$

(ii) for any cone $\Theta' \subset\subset \Theta$ and any integer N we have (2.12)

$$|\partial_x^\alpha \partial_\xi^\beta (f_{(\lambda)} - g_{(\lambda)})(x,\xi)| \leq C_{\alpha\beta N\Theta'}(x)(1+|\lambda|)^p \|\xi\|^{-N}, \qquad \xi \neq 0, \quad \lambda \in \Theta'.$$

The homogeneity of $g_{(\lambda)}$ implies that for any cone $\Theta' \subset\subset \Theta$ we have

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta g_{(\lambda)}(x,\xi)| &= \|\xi\|^{m-\langle\beta\rangle} |g_{(x,\|\xi\|^{-2}\lambda)}(\|\xi\|^{-1}.\xi)| \\ &\leq C_{\alpha\beta\Theta'}(x)\|\xi\|^{m-\langle\beta\rangle}(1+\|\xi\|^{-2}|\lambda|)^p, \end{aligned} \tag{2.13}$$

for $\xi \neq 0$ and $\lambda \in \Theta'$. Since

$$\|\xi\|^{-2}(1+|\lambda|) \leq 1 + \|\xi\|^{-2}|\lambda| \leq 1 + |\lambda|, \qquad \|\xi\| > 1, \quad \lambda \in \Theta, \tag{2.14}$$

the estimates (2.9) for $\lambda \in \Theta$ follow from (2.12) and (2.13).

Now, let us prove (2.9) for $\lambda \in D$. Since D is bounded it is enough to show that we have

$$|\partial_x^\alpha \partial_\xi^\beta f_{(\lambda)}(x,\xi)| \leq C_{\alpha\beta}(x,\lambda)(1+\|\xi\|)^{m+q-\langle\beta\rangle}, \qquad \lambda \in D, \tag{2.15}$$

with $C_{\alpha\beta}(x,\lambda)$ locally bounded on $U \times D$. The definition of $S^p_m(U \times \mathbb{R}^{d+1}, \Lambda)$ implies that for any integer N we have (2.16)

$$|2^{-m} f_{(\lambda)}(x, 2.\xi) - f_{(4^{-1}\lambda)}(x,\xi)| \leq C_{\Lambda' N}(x,\lambda)\|\xi\|^{-N}, \qquad \xi \neq 0, \ \lambda \in D.$$

If N is taken sufficiently large for $\xi \neq 0$ and $\lambda \in D'$ we get

$$\begin{aligned} &|2^{-km} f_{(\lambda)}(x, 2^k.\xi) - f_{(4^{-k}\lambda)}(x,\xi)| \\ \leq\ & \sum_{j=0}^{k-1} |2^{-jm}(2^{-m} f_{(4^{j+1-k}\lambda)}(x, 2^{j+1}.\xi,) - f_{(4^{j-k}\lambda)}(x, 2^{-jm}.\xi))| \\ \leq\ & C_N(x,\lambda)\|\xi\|^{-N}. \end{aligned} \tag{2.17}$$

Let $\Gamma = \{\xi \in \mathbb{R}^{d+1}; 1 < \|\xi\| \leq 2\}$. As Γ is compact we obtain

$$|f_{(\lambda)}(2^k.\xi)| \leq C(x,\lambda)2^{mk}, \qquad \xi \in \Gamma, \quad \lambda \in D. \tag{2.18}$$

Since each $\xi \in \mathbb{R}^{d+1}$ with $\|\xi\| > 1$ is of the form $\xi = 2^k\eta$ with $\eta \in \Gamma$ and $\frac{1}{2}\|\xi\| \leq 2^k < \|\xi\|$, we conclude that

$$|f_{(\lambda)}(\xi)| \leq C(x,\lambda)\|\xi\|^m, \qquad \|\xi\| > 1, \quad \lambda \in D. \tag{2.19}$$

This gives the estimate (2.9) for $\lambda \in D$ in the case $\alpha = \beta = 0$. The estimates for $\alpha \neq 0$ and $\beta \neq 0$ are obtained similarly. ■

It is then possible to make the following definition:

Definition 2.7 *$S^{p,m}(U \times \mathbb{R}^{d+1}, \Lambda)$, $m, p \in \mathbb{Z}$, is the space of* $\mathrm{Hol}^p(\Lambda)$*- families $f_{(\lambda)}$ of smooth functions on $U \times \mathbb{R}^{d+1}$ with an asymptotic expansion*

$$f_{(\lambda)} \sim \sum_{j \geq 0} f_{(\lambda),m-j}, \qquad f_{(\lambda),k} \in S^p_k(U \times \mathbb{R}^{d+1}, \Lambda), \tag{2.20}$$

in the sense that, for any integer N, if $J$is large enough we have

$$f - \sum_{j \leq J} f_{(\lambda),m-j} \in S^{p,-N}_{||}(U \times \mathbb{R}^{d+1}, \Lambda). \tag{2.21}$$

Note that the asymptotic expansion 2.20 determines $f_{(\lambda)}$ up to an element of $S^{p,-\infty}(U \times \mathbb{R}^{d+1}, \Lambda)$. Conversely by [Ho1, theorem 2.7] we have

Proposition 2.8 *Suppose given for $j = 0, 1, \ldots$ some $f_{(\lambda),m-j} \in S^{p,m-j}(U \times \mathbb{R}^{d+1}, \Lambda)$. Then there exists $f_{(\lambda)}$ in $S^{p,m}(U \times \mathbb{R}^{d+1}, \Lambda)$ such that $f_{(\lambda)} \sim \sum f_{(\lambda),m-j}$.*

2.2 Parametric $\Psi_{\mathcal{V}}DO$ operators on an open subset of $\mathbb{R}^{d+1}$

Definition 2.9 *For $p \in \mathbb{Z}$ we denote by $\Psi^{p,-\infty}_{\mathcal{V}}(U, \Lambda)$ the space consisting in the families of operators given by a $Hol^p(\Lambda)$-family of smooth kernels.*

Definition 2.10 *For $m, p \in \mathbb{Z}$ the space $\Psi^{p,m}_{\mathcal{V}}(U, \Lambda)$ consists in families $P_{(\lambda)}$ with values in $\mathcal{L}(C^\infty_c(U), C^\infty(U))$ of the form*

$$P_{(\lambda)} = f_{(\lambda)}(x, \sigma(x, D)) + R_{(\lambda)}, \tag{2.22}$$

where $f_{(\lambda)}$ is in $S^{p,m}(U \times \mathbb{R}^{d+1}, \Lambda)$, called the symbol of $P_{(\lambda)}$, and $R_{(\lambda)}$ is a $\mathrm{Hol}^p(\Lambda)$*-family of smoothing operators.*

Proposition 2.11 *Let $m, p \in \mathbb{Z}$ and set $k = m + q$ if $m + q \geq 0$ and $k = \frac{1}{2}(m + q)$ otherwise.*

a) The class $\Psi_{\mathcal{V}}^{p,m}(U, \Lambda)$ does not depend on the choice of the $\mathcal{V}$-frame $X_0, X_1, \ldots, X_d$.

b) Each $P_{(\lambda)} \in \Psi_{\mathcal{V}}^{p,m}(U, \Lambda)$ extends to a $\mathrm{Hol}^p(\Lambda)$-family of continuous operators from $\mathcal{E}'(U)$ into $\mathcal{D}'(U)$.

c) The kernel of any $P_{(\lambda)} \in \Psi_{\mathcal{V}}^{p,m}(U, \Lambda)$ is given outside the diagonal Δ of $U \times U$ by a $\mathrm{Hol}^p(\Lambda)$-family of smooth functions.

d) For any $s \in \mathbb{R}$, each $P_{(\lambda)} \in \Psi_{\mathcal{V}}^{p,m}(U, \Lambda)$ defines a $\mathrm{Hol}^p(\Lambda)$-family of continuous operators from $H^s_{\mathrm{comp}}(U)$ into $H^{s-k}_{\mathrm{loc}}(U)$.

Proof. 1) The proof of the independence with respect to the $\mathcal{V}$-frame follows along the same lines of the proof in the non parameter case (see [BG, proposition 10.46]). It will be also a consequence of the proof of the invariance by Heisenberg diffeomorphisms (proposition 2.21).

2) By [BG, proposition 10.22] and the closed graph theorem the map $f(x, \xi) \longrightarrow f(x, \sigma(x, D))$ is continuous from $S_{||}^{m+q}(U \times \mathbb{R}^{d+1})$ into $S^k_{\frac{1}{2},\frac{1}{2}}(U \times \mathbb{R}^{d+1})$. Moreover it follows from [Ho1, theorem 2.2] that the quantization map $q \longrightarrow q(x, D)$ is continuous from $S^k_{\frac{1}{2},\frac{1}{2}}(U \times \mathbb{R}^{d+1})$ into $\mathcal{L}(\mathcal{E}'(U), \mathcal{D}'(U))$. Therefore each $P_{(\lambda)} \in \Psi_{\mathcal{V}}^{p,m}(U, \Lambda)$ extends to a $\mathrm{Hol}^p(\Lambda)$-family of continuous operators from $\mathcal{E}'(U)$ into $\mathcal{D}'(U)$.

3) The continuity of the above quantization map and the closed graph theorem implies that the map which assigns to $q \in S^k_{\frac{1}{2},\frac{1}{2}}(U \times \mathbb{R}^{d+1})$ the restriction of the kernel of $q(x, D)$ on $U \times U \setminus \Delta$ is continuous from $S^k_{\frac{1}{2},\frac{1}{2}}(U \times \mathbb{R}^{d+1})$ into $C^\infty(U \times U \setminus \Delta)$. It follows that the kernel of any $P_{(\lambda)} \in \Psi_{\mathcal{V}}^{p,m}(U, \Lambda)$ is given outside the diagonal by a $\mathrm{Hol}^p(\Lambda)$-family of smooth functions.

4) Let $s \in \mathbb{R}$ and $P_{(\lambda)} \in \Psi_{\mathcal{V}}^{p,m}(U, \Lambda)$. By [Hw, theorem 3] the map $q \to q(x, D)$ is continuous from $S^k_{\frac{1}{2},\frac{1}{2}}(U \times \mathbb{R}^{d+1})$ into $\mathcal{L}(H^s_{\mathrm{comp}}, H^{s-k}_{\mathrm{loc}})$. So the the same arguments as above show that $P_{(\lambda)}$ defines a $\mathrm{Hol}^p(\Lambda)$-family of continuous operators from $H^s_{\mathrm{comp}}(U)$ into $H^{s-k}_{\mathrm{loc}}(U)$. ■

Definition 2.12 *A parametric operator from $P_{(\lambda)} : \mathcal{E}'(U) \to \mathcal{D}'(U)$ is said uniformly properly supported if its kernel is properly supported uniformly with respect to λ.*

Proposition 2.13 *Let $P_{(\lambda)} \in \Psi_{\mathcal{V}}^{p,m}(U, \Lambda)$. Then:*

1) We can write $P_{(\lambda)}$ as $P_{(\lambda)} = Q_{(\lambda)} + R_{(\lambda)}$ with $Q_{(\lambda)} \in \Psi_{\mathcal{V}}^{p,m}(U,\Lambda)$ uniformly properly supported and $R_{(\lambda)}$ a $\mathrm{Hol}^p(\Lambda)$-family of smoothing operators.

2) If $P_{(\lambda)}$ is uniformly properly supported, it defines $\mathrm{Hol}^p(\Lambda)$-family of continuous endomorphisms of respectively $C_c^\infty(U)$, $C^\infty(U)$, $\mathcal{E}'(U)$ and $\mathcal{D}'(U)$.

Proof. 1) Pick some $\chi \in C_c(U-U)$ such that $\chi = 1$ near 0 and let $k_{(\lambda)}(x,y)$ be the kernel of $P_{(\lambda)}$. Then the property is satisfied by $Q_{(\lambda)}$ and $R_{(\lambda)}$ with respective kernels $\chi(x-y)k_{(\lambda)}(x,y)$ and $(1-\chi(x-y))k_{(\lambda)}(x,y)$.

2) If $P_{(\lambda)}$ is uniformly properly supported it is clear that it defines a $\mathrm{Hol}^p(\Lambda)$-family of continuous endomorphisms of $C_c^\infty(U)$ and $\mathcal{E}'(U)$. As the same is true for the transpose $P_{(\lambda)}^t$ the remainder of the assertion follows by duality. ■

Let us now look at the composition of $\Psi_{\mathcal{V}}DO$ operators with parameter. By continuity the convolution $*$ for symbols gives rise to a bilinear map

(2.23)
$$S_{||}^{p_1,k_1}(U\times\mathbb{R}^{d+1},\Lambda)\times S_{||}^{p_2,k_2}(U\times\mathbb{R}^{d+1},\Lambda)\longrightarrow S_{||}^{p_1+p_2,k_1+k_2}(U\times\mathbb{R}^{d+1},\Lambda).$$

As $S^{p,-\infty}(U\times\mathbb{R}^{d+1},\Lambda)$ is a two-sided ideal we get a convolution on $S_*^p(U\times\mathbb{R}^{d+1},\Lambda)$ as a bilinear map

$$(2.24)\qquad *: S_{m_1}^{p_1}(U\times\mathbb{R}^{d+1})\times S_{m_2}^{p_2}(U\times\mathbb{R}^{d+1})\longrightarrow S_{m_1+m_2}^{p_1+p_2}(U\times\mathbb{R}^{d+1}).$$

To state the composition formula we keep the notations of proposition 1.18.

Proposition 2.14 *Let $P_{i(\lambda)} \in \Psi_{\mathcal{V}}^{p_i,m_i}(U,\Lambda)$, $i=1,2$, with symbol $f_{i(\lambda)} \sim \sum f_{i(\lambda),m_1-j}$ and suppose either $P_{1(\lambda)}$ or $P_{2(\lambda)}$ uniformly properly supported. Then $P_{1(\lambda)}P_{2(\lambda)}$ lies in $\Psi_{\mathcal{V}}^{p_1+p_2,m_1+m_2}(U,\Lambda)$ and has symbol $f_{(\lambda)} \sim \sum f_{(\lambda),m_1+m_2-j}$ with*

$$(2.25)\qquad f_{(\lambda),m_1+m_2-j} = \sum h_{\alpha\beta\gamma\delta}(x) f^{\delta}_{1(\lambda),m_1-k} * f^{\beta\gamma}_{2(\lambda),m_2-l,\alpha}(x,\xi),$$

where the summation is taken over the indices such that $|\beta|+|\alpha| \leq \langle\delta\rangle + \langle\beta\rangle - \langle\gamma\rangle = j-k-l$ and $|\gamma| = |\beta|$.

Proof. As either $P_{1(\lambda)}$ or $P_{2(\lambda)}$ is uniformly properly supported the proposition 2.13 allows us to suppose both $P_{1(\lambda)}$ and $P_{(\lambda)}$ uniformly properly supported. Then up to a $\mathrm{Hol}^p(\Lambda)$-family of smoothing operators we have

$$(2.26)\quad P_{1(\lambda)}P_{2(\lambda)} = \sum_i \varphi_i P_{1(\lambda)}\psi_i P_{2(\lambda)} = \sum \varphi_i f_{1(\lambda)})\#_{\psi_i} f_{2(\lambda)}(x,\sigma(x,D)),$$

where (φ_i) is a locally finite partition of unity and $(\psi_i) \subset C_c^\infty(U)$ is such that $\psi_i = 1$ near $\operatorname{supp} \varphi_i$.

However it follows from lemma 1.19 and remark 1.22 that for a given $\psi \in C_c^\infty(U)$ the family $f_{1(\lambda)})\#_\psi f_{2(\lambda)}$ is a $\mathrm{Hol}^p(\Lambda)$-family of smooth functions with an asymptotic expansion $f_{1(\lambda)})\#_\psi f_{2(\lambda)} \sim \sum h_{\alpha\beta\gamma\delta} f_{1(\lambda)}^{\delta} * f_{2(\lambda),\alpha}^{\beta\gamma}$ in the sense of $\mathrm{Hol}^p(\Lambda)$-families of symbols. Therefore $P_{1(\lambda)} P_{2(\lambda)}$ lies in $\Psi_{\mathcal{V}}^{p_1+p_2, m_1+m_2}(U, \Lambda)$ and has symbol $f_{(\lambda)} \sim \sum f_{(\lambda), m_1+m_2-j}$ with $f_{(\lambda), m_1+m_2-j}$ given by (2.25). ■

2.3 Kernels of $\Psi_{\mathcal{V}} DO$ operators with parameter

Let us now study the kernels of $\Psi_{\mathcal{V}} DO$ operators with parameter.

Definition 2.15 *$\mathcal{K}_m^p(U \times \mathbb{R}^{d+1}, \Lambda)$, $m, p \in \mathbb{Z}$, consists in $\mathrm{Hol}^p(\Lambda)$-families $K_{(\lambda)}$ with values in $C^\infty(U) \hat{\otimes} \mathcal{D}'(\mathbb{R}^{d+1})$ such that:*

(i) restricted to $U \times (\mathbb{R}^{d+1} \backslash 0)$ the family $K_{(\lambda)}$ is given by $\mathrm{Hol}^p(\Lambda)$-family of smooth functions;

(ii) for any $t > 1$ the family $K_{(t^{-2}\lambda)}(x, t.y) - t^m K_{(\lambda)}(x, y)$ is a $\mathrm{Hol}^p(\Lambda)$-family of smooth functions.

The arguments in the proof of lemma 1.32 show that we have the following characterization of almost homogeneous kernels with parameter.

Lemma 2.16 *Let $m, p \in \mathbb{Z}$ and set $\hat{m} = -(m + d + 2)$.*

a) If $f_{(\lambda)}(x, \xi) \in S_m^p(U \times \mathbb{R}^{d+1}, \Lambda)$ then $\check{f}_{(\lambda)\xi \to y}(x, y)$ belongs to $\mathcal{K}_{\hat{m}}^p(U \times \mathbb{R}^{d+1}, \Lambda)$.

b) If $K_{(\lambda)}(x, y) \in \mathcal{K}_{\hat{m}}^p(U \times \mathbb{R}^{d+1}, \Lambda)$ is compactly supported in y uniformly with respect to λ, then $\hat{K}_{(\lambda)y \to \xi}(x, \xi)$ belongs to $S_m^p(U \times \mathbb{R}^{d+1}, \Lambda)$.

Definition 2.17 *$\mathcal{K}^{p,m}(U \times \mathbb{R}^{d+1}, \Lambda)$, $m, p \in \mathbb{Z}$, is the space of families $K_{(\lambda)}$ of distributions on $U \times \mathbb{R}^{d+1}$ with an asymptotic expansion*

$$K_{(\lambda)} \sim \sum_{j \geq 0} K_{(\lambda), m+j}, \qquad K_{(\lambda), m+j} \in \mathcal{K}_{m+j}^p(U \times \mathbb{R}^{d+1}, \Lambda), \tag{2.27}$$

where $\sim$ means that for any integer N if J is large enough we have

$$K_{(\lambda)} - \sum_{j \leq J} K_{(\lambda), m+j} \in \mathrm{Hol}^p(\Lambda, C^N(U \times \mathbb{R}^{d+1})). \tag{2.28}$$

Remark 2.18 The definition implies that $K_{(\lambda)}$ lies in $\mathrm{Hol}^p(\Lambda, C^\infty(U) \hat{\otimes} \mathcal{D}(\mathbb{R}^{d+1})')$ and is given on $U \times (\mathbb{R}^{d+1} \backslash 0)$ by a $\mathrm{Hol}^p(\Lambda)$-family of smooth functions.

We can now characterize $\Psi_{\mathcal{V}}DO$'s with parameter.

Proposition 2.19 *Let $P_{(\lambda)}$ be a family of continuous operators from $C_c^\infty(U)$ into $C^\infty(U)$. Then $P_{(\lambda)}$ lies in $\Psi_{\mathcal{V}}^{p,m}(U,\Lambda)$ if, and only if, its kernel is of the form*

$$k_{(\lambda)}(x,y) = |\varepsilon_x'| K_{(\lambda)}(x, -\varepsilon_x(y)) + R_{(\lambda)}(x,y), \tag{2.29}$$

for some $K_{(\lambda)} \in \mathcal{K}^{p,\hat{m}}(U \times \mathbb{R}^{d+1}, \Lambda)$, $\hat{m} = -(m+d+2)$, and some $\mathrm{Hol}^p(\Lambda)$*-family $R_{(\lambda)}$ of smoothing operators.*

Proof. As in the proof of proposition 1.29 we need only to check that a family $K_{(\lambda)}$ of distributions on $U \times \mathbb{R}^{d+1}$ lies in $\mathcal{K}^{p,\hat{m}}(U \times \mathbb{R}^{d+1}, \Lambda)$ if , and only if, it is of the form

$$K_{(\lambda)}(x,y) = \check{f}_{(\lambda)\xi\to y}(x,y) + R_{(\lambda)}(x,y), \tag{2.30}$$

for some $f_{(\lambda)} \in S^{\hat{m},p}(U \times \mathbb{R}^{d+1}, \Lambda)$ and some $\mathrm{Hol}^p(\Lambda)$-family $R_{(\lambda)}$ of smooth functions. However the Fourier transform draws equivalences between:

(i) the classes $S_m^p(U \times \mathbb{R}^{d+1}, \Lambda)$ and $\mathcal{K}_{\hat{m}}^p(U \times \mathbb{R}^{d+1}, \Lambda)$ up to $\mathrm{Hol}^p(\Lambda)$-families of smooth functions;

(ii) the asymptotic expansions for $\mathrm{Hol}^p(\Lambda)$-families of symbols and $\mathrm{Hol}^p(\Lambda)$-families of kernels.

Then arguing as in the proof of proposition 1.29 we reach the conclusion. ■

2.4 Invariance by Heisenberg diffeomorphisms

Before proving the invariance by Heisenberg diffeomorphism we need the following lemma.

Lemma 2.20 *Let $k \in \mathbb{R}$ and let μ be the smallest positive integer $> \frac{1}{2}(k + d + 2)$. Then for any $f \in S_{||}^k(U \times \mathbb{R}^{d+1})$ we have*

$$|\partial_x^\alpha \check{f}_{\xi\to y}(x,y)| \leq C_{k\alpha}(x,f)\|y\|^{-\mu}, \qquad 0 < \|y\| \leq 1, \tag{2.31}$$

with $C_{k\alpha}(x,f)$ locally bounded on $U \times S_{||}^k(U \times \mathbb{R}^{d+1})$.

Proof. Let $f \in S_{||}^k(U \times \mathbb{R}^{d+1})$. In the case $k < -(d+2)$ we have

$$|\partial_x^\alpha \check{f}_{\xi\to y}(x,y)| \leq C_{k\alpha}(x,f) = \int |f(x,\xi)| d\xi. \tag{2.32}$$

Suppose now $k \geq -(d+2)$. Applying the above inequality to $\partial_{\xi_0}^{\mu} f$ and $\partial_{\xi_j}^{2\mu} f$, $1 \leq j \leq d$, we get

$$(|y_0|^{\mu} + \sum_{j=1}^{d} |y_j|^{2\mu})\ |\partial_x^{\alpha} \check{f}_{\xi\to y}(x,y)| \leq C_{k\alpha}(x,f), \tag{2.33}$$

with $C_{k\alpha}(x,f)$ locally bounded on $U \times S_{||}^{k}(U \times \mathbb{R}^{d+1})$. This completes the proof. ■

Proposition 2.21 *Let $\phi : U \to \tilde{U}$ be a Heisenberg diffeomorphism where $\tilde{U}$ is another subset of $\mathbb{R}^{d+1}$ equipped with a hyperplane bundle $\tilde{\mathcal{V}} \subset T\tilde{U}$ and a $\tilde{\mathcal{V}}$-frame. Then for any $\tilde{P}_{(\lambda)} \in \Psi_{\tilde{\mathcal{V}}}^{p,m}(\tilde{U},\Lambda)$ the pullback $P_{(\lambda)} = \phi^* \tilde{P}_{(\lambda)}$ lies in $\Psi_{\mathcal{V}}^{p,m}(U,\Lambda)$.*

Proof. Let $K_{(\lambda)} \in \mathcal{K}^{\hat{m}}(U \times \mathbb{R}^{d+1})$. By lemma 2.20 there exists an integer $\mu \geq 0$ such that for any pseudocone $\Lambda' \subset\subset \Lambda$ and $\langle\beta\rangle$ large enough we have

$$|\partial_x^{\alpha}\partial_y^{\beta} K_{(\lambda)}| \leq C_{\alpha\beta\Lambda'}(x)\|y\|^{-\mu-\langle\beta\rangle}, \qquad 0 < \|y\| \leq 1, \quad \lambda \in \Lambda', \tag{2.34}$$

with $C_{\alpha\beta\Lambda'}(x)$ locally bounded on U. This remark enables the arguments of the proof of proposition 1.33 to work *mutatis standis* for kernels with parameter and therefore to prove the invariance by Heisenberg diffeomorphisms. ■

Remark 2.22 In the special case $U = \tilde{U}$, $\mathcal{V} = \tilde{\mathcal{V}}$ and $\phi = \tilde{\phi}$ we obtain the independence with respect to the $\mathcal{V}$-frame.

2.5 Parametric $\Psi_{\mathcal{V}}DO$'s on manifolds

The proposition 2.21 enables us to define $\Psi_{\mathcal{V}}DO$'s with parameter on any Heisenberg manifold.

Definition 2.23 *Let $(M,\mathcal{V})$ be a Heisenberg manifold. Then $\Psi_{\mathcal{V}}^{p,m}(M,\Lambda)$, $m,p \in \mathbb{Z}$, is the space of families $P_{(\lambda)} \in \mathrm{Hol}^p(\Lambda, \mathcal{L}(C_c^{\infty}(M), C^{\infty}(M))$ such that:*

(i) for φ, ψ in $C^{\infty}(M)$ the family $\varphi P_{(\lambda)} \psi$ is a $\mathrm{Hol}^p(\Lambda)$ of smoothing operators;

(ii) on any Heisenberg chart $P_{(\lambda)}$ is given by a parametric $\Psi_{\mathcal{V}}DO$ in $\Psi_{\mathcal{V}}^{p,m}(U,\Lambda)$ where U is an open subset of $\mathbb{R}^{d+1}$ with a $\mathcal{V}$-frame.

Remark 2.24 There is a similar definition for $\Psi_{\mathcal{V}}DO$'s with parameter acting on sections of a vector bundle over a Heisenberg manifold.

All the results of the previous sections continue to hold on a general Heisenberg manifold. In particular for a compact Heisenberg manifold we get:

Proposition 2.25 *Let $(M,\mathcal{V})$ be a compact Heisenberg manifold.*

1) Let $m_1, m_2, p_1, p_2 \in \mathbb{Z}$. Then for any $P_{1(\lambda)} \in \Psi_{\mathcal{V}}^{p_1,m_1}(M,\Lambda)$ and $P_{1(\lambda)} \in \Psi_{\mathcal{V}}^{p_2,m_2}(M,\Lambda)$ the family $P_{1(\lambda)}P_{1(\lambda)}$ lies in $\Psi_{\mathcal{V}}^{p_1+p_2,m_1+m_2}(M,\Lambda)$.

2) Let $m,p \in \mathbb{Z}$. Then any $P_{(\lambda)} \in \Psi_{\mathcal{V}}^{p,m}(M,\Lambda)$ extends to a family in $\mathrm{Hol}^p(\Lambda, \mathcal{L}(H^s(M), H^{s-k}(M)))$, $s \in \mathbb{R}$, where, with $q = 2\max(0,-p)$, k is equal to $m+q$ if $m+q \geq 0$ and to $\frac{1}{2}(m+q)$ otherwise.

2.6 Asymptotic resolvent for sublaplacians

Let us now construct an asymptotic resolvent for an elliptic sublaplacian Δ as a parametrix for $\Delta - \lambda$ in the parametric $\Psi_{\mathcal{V}}DO$ calculus. For achieving that we shall set

$$\Lambda_0 = \{\lambda \in \mathbb{C}\backslash 0;\ \Re\lambda < 0\}, \tag{2.35}$$

$$\Lambda_R = \{\lambda \in \mathbb{C}\backslash 0;\ \Re\lambda < 0 \text{ or } |\lambda| < R\}, \qquad R > 0. \tag{2.36}$$

Proposition 2.26 *Let Δ be an elliptic sublaplacian on U in the form*

$$\Delta = -\sum_{j=1}^{d} X_j^2 - i\nu(x)X_0 + \sum_{j=1}^{d} \mu_j(x)X_j + \eta(x), \tag{2.37}$$

where ν, $\mu_1, \ldots, \mu_d$, η are smooth functions. Let $p_2(x,\xi) = \sum_{j=1}^{d} \xi_j^2 + i\nu(x)\xi_0$ be the principal symbol of Δ. Then for any $R > 0$ there exists $f_{(\lambda)} \in S_{-2}^{-1}(U \times \mathbb{R}^{d+1}, \Lambda_R)$ such that

$$(p_2 - \lambda) * f_{(\lambda)} = 1 = f_{(\lambda)} * (p_2 - \lambda) \qquad \mathrm{mod}\ S^{-\infty,-\infty}(U \times \mathbb{R}^{d+1}, \Lambda_R). \tag{2.38}$$

Proof. The proof follows closely [BG, chapters 1–2] and the proof of theorem 5.22 in [BGS]. Let us first precise the subellipticity condition for Δ. Consider the 1-form θ annihilating $\mathcal{V}$ such that $\theta(X_0) = 1$ and let L be the Hermitian function-valued form on $\mathcal{V}$ defined by

$$L(X,Y) = -id\theta(X,Y) = i\theta([X,Y]), \qquad X, Y \in \mathcal{V}. \tag{2.39}$$

Note that $L(X,Y)$ is characterized by

$$[X,Y] = -iL(X,Y) \qquad \mathrm{mod}\ \mathcal{V}. \tag{2.40}$$

At $y \in U$ the Hermitian function-valued form L defines a Hermitian form L_y which is purely imaginary and has real eigenvalues $a_1 = a_1(y), \ldots, a_d = a_d(y)$ so that $a_j \geq 0$ and $a_{n+j} = -a_j$ for $j = 1, \ldots, n$, and $a_j = 0$ for $j = 2n+1, \ldots, d$. The ellipticity condition for Δ is

$$|\Re v(y)| < \frac{1}{2} \sum_{j=1}^{d} |a_j(y)|. \tag{2.41}$$

For $y \in U$ denote by p_2^y and $f_{(\lambda)}^y$ the symbols $p_2(y, .)$ and $f_{(\lambda)}(y, .)$. Then we need only to find for each $y \in U$ a symbol $f_{(\lambda)}^y \in S_{-2}^{-1}(\mathbb{R}^{d+1}, \Lambda_R)$ such that

$$(p_2^y - \lambda) *^y f_{(\lambda)}^y = 1 = f_{(\lambda)}^y *^y (p_2^y - \lambda) \qquad \mod S^{-\infty,-\infty}(\mathbb{R}^{d+1}, \Lambda_R), \tag{2.42}$$

and in such way that everything is smooth with respect to y, i.e.

(i) the symbol $f_{(\lambda)}(y, \xi) = f_{(\lambda)}^y(\xi)$ lies in $S_{-2}^{-1}(U \times \mathbb{R}^{d+1}, \Lambda_R)$;

(ii) the remainder terms $r_{1(\lambda)}(y, \xi) = 1 - (p_2^y - \lambda) *^y f_{(\lambda)}^y$ and $r_{2(\lambda)}(y, \xi) = 1 - f_{(\lambda)}^y *^y (p_2^y - \lambda)$ belong to $S^{-\infty,-\infty}(U \times \mathbb{R}^{d+1}, \Lambda_R)$.

This will be achieved in 3 steps:

1) Find for each $y \in U$ coordinates called the normal y-coordinates in which the equations (2.42) take a simple form.

2) Resolution of (2.42) in the normal y-coordinates.

3) Return to the original y-coordinates and show that the resulting symbol satisfies to the above conditions (i) and (ii).

Step 1: construction of the normal y-coordinates (*cf.* §1 of [BG]). In the y-coordinates relatively to this $\mathcal{V}$-frame we have

$$X_j = \frac{\partial}{\partial x_j} + \sum_{j=0}^{d} \gamma_{jk}(x) \frac{\partial}{\partial x_k}, \qquad \gamma_{jk}(0) = 0. \tag{2.43}$$

The y-invariant vector fields are

$$X_0^y = \frac{\partial}{\partial x_0}, \tag{2.44}$$

$$X_j = \frac{\partial}{\partial x_j} + \sum_{j=0}^{d} c_{jk} x_k \frac{\partial}{\partial x_0}, \qquad 1 \leq j \leq d, \tag{2.45}$$

with $c_{jk} = c_{jk}(y) = \frac{\partial}{\partial x_k}\gamma_{jk}(0)$. The change of coordinates

$$(x_0, x') \to (x_0 - \frac{1}{4}\sum_{j,k=1}^{d}(c_{jk} + c_{kj})x_j x_k, x'), \tag{2.46}$$

brings the matrix (c_{jk}) into anti-symmetric form. Namely the vector fields X_j^y have the same expression with c_{jk} replaced by

$$a_{jk} = \frac{1}{2}(c_{jk} - c_{kj}). \tag{2.47}$$

We refer to these coordinates as the *anti-symmetric y-coordinates.*

However we have

$$L_y(X_j, X_k) = i\theta_y([X_j, X_k]) = \frac{i}{2}(c_{kj} - c_{jk}) = -ia_{jk}. \tag{2.48}$$

Thus there exists an orthogonal matrix Q which brings the matrix $A = (a_{jk})$ into the normal form

$$Q^t A Q = \begin{pmatrix} 0 & -A' & 0 \\ A' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A' = \operatorname{diag}(a_1, \cdots, a_n), \tag{2.49}$$

where $a_1, \cdots, a_n$ are the nonnegative eigenvalues of L_y. Then making the orthogonal change of coordinates

$$(x^0, x') \longrightarrow (x^0, Q^t x') \tag{2.50}$$

we put the vector fields X_j^y into the form

$$X_0^y = \frac{\partial}{\partial x^0}; \qquad X_j^y = \frac{\partial}{\partial x^j} - \frac{1}{2}a_j x^{n+j}\frac{\partial}{\partial x^0}, \tag{2.51}$$

$$X_{n+j}^y = \frac{\partial}{\partial x^{n+j}} + \frac{1}{2}a_j x^j \frac{\partial}{\partial x^0}, \qquad 1 \leq j \leq n; \tag{2.52}$$

$$X_k^y = \frac{\partial}{\partial x^k}, \qquad k = 2n < k \leq d. \tag{2.53}$$

We refer to these coordinates as *normal y-coordinates.*

To end up with this step note that we can define the anti-symmetric and normal y-groups as the 2-step nilpotent groups associated to the y-invariant vector fields X_j^y in the anti-symmetric and normal y-coordinates. Then the changes of coordinates (2.46) and (2.50) becomes isomorphisms from the original y-group onto the anti-symmetric one and from the anti-symmetric

y-group onto the normal one.

Step 2: resolution in the normal y-coordinates. Let

$$\Delta_2^y = p_2^y(\sigma^y(x,D)) = -\sum_{j=1}^{d}(X_j^y)^2 - i\nu X_0, \qquad \nu = \nu(y). \tag{2.54}$$

By the very definition of $*^y$ the symbol $(p_2^y - \lambda) *^y f_{(\lambda)}$ is characterized by the equality

$$(p_2^y - \lambda) *^y f_{(\lambda)}(\sigma^y(x,D)) = (\Delta_2^y - \lambda) f_{(\lambda)}(\sigma^y(x,D)). \tag{2.55}$$

In other words $(p_2^y - \lambda) *^y f_{(\lambda)}$ is the symbol at $x = 0$ of the y-invariant operator

$$(\Delta_2^y - \lambda) f_{(\lambda)}(\sigma^y(x,D)). \tag{2.56}$$

In the normal y-coordinates Δ_2^y has standard symbol
(2.57)

$$q_2^y(x,\xi) = \sum_{j=1}^{n}(\xi_j - \frac{1}{2}x^{n+j}a_j\xi_0)^2 + \sum_{j=1}^{n}(\xi_{n+j} + \frac{1}{2}x^j a_j \xi_0)^2 + \sum_{j=n+1}^{d}\xi_j^2 + \nu\xi_0.$$

As Δ_2^y is a differential operator the standard symbol of the left-hand side in (2.55) can be explicitly calculated. Letting $q_{(\lambda)}^y(x,\xi) = f_{(\lambda)}(y, \sigma^y(x,\xi))$ it yields

$$\begin{aligned}
(p_2^y - \lambda) *^y f_{(\lambda)}(\xi) &= \sum_{|\alpha|\leq 2} \frac{1}{\alpha!}\partial_\xi^\alpha q_2^y(0,\xi) D_x^\alpha q_{(\lambda)}^y(0,\xi) - \lambda f_{(\lambda)}(\xi), \\
&= (\sum_{j=1}^{d}\xi_j^2 + \nu - \lambda) f_{(\lambda)}(\xi) + \frac{1}{i}\xi_0 \sum_{j=1}^{n}(\xi_{n+j}\partial_{\xi_j} - \xi_j \partial_{\xi_{n+j}}) f_{(\lambda)}(\xi) \\
&\quad - \sum_{j=1}^{2n} a_j^2 \xi_0^2 \partial_{\xi_j}^2 f_{(\lambda)}(\xi).
\end{aligned} \tag{2.58}$$

Similarly we can compute $f_{(\lambda)}(\xi) *^y (p_2^y - \lambda)$ and find

$$\begin{aligned}
f_{(\lambda)}(\xi) *^y (p_2^y - \lambda) &= (\sum_{j=1}^{d}\xi_j^2 + \nu - \lambda) f_{(\lambda)}(\xi) + \frac{1}{i}\xi_0 \sum_{j=1}^{n}(\xi_j \partial_{\xi_{n+j}} - \xi_{n+j}\partial_{\xi_j}) f_{(\lambda)}(\xi) \\
&\quad -\frac{1}{4}\sum_{j=1}^{2n} a_j^2 \xi_0^2 \partial_{\xi_j}^2 f_{(\lambda)}(\xi).
\end{aligned} \tag{2.59}$$

However the form of Δ_2^y in the normal y-coordinates shows that it is invariant under rotations in the (x_j, x_{n+j}) plane. We can require the same to hold for $q_{(\lambda)}(x, D)$ which is equivalent to the invariance of $f_{(\lambda)}$ under rotations in the (ξ_j, ξ_{n+j}) plane, i.e.

$$\xi_j \frac{\partial}{\partial \xi_{n+j}} f_{(\lambda)}(\xi, \lambda) - \xi_{n+j} \frac{\partial}{\partial \xi_j} f_{(\lambda)}(\xi, \lambda) \qquad j = 1, \dots, n. \tag{2.60}$$

Therefore we need only to find a solution invariant under rotations in the (ξ_j, ξ_{n+j}) planes, $j = 1, \dots, n$, of the single equation
(2.61)

$$\sum_{j=1}^{d} (\xi_j^2 - \frac{1}{4} a_j^2 \xi_0^2 \partial_{\xi_j}^2) f_{(\lambda)} + (\nu\xi_0 - \lambda) f_{(\lambda)} = 0 \qquad \bmod S^{-\infty,-\infty}(\mathbb{R}^{d+1}, \Lambda_R).$$

This yields the formal solution:

$$f_{(\lambda)}(\xi) = \int_0^\infty e^{(\lambda - \nu\xi_0)s} G(\xi, s) ds, \tag{2.62}$$

where $G(\xi, s)$ satisfies $\sum_{j=1}^{d} (\xi_j^2 - \frac{1}{4} a_j^2 \xi_0^2 \partial_{\xi_j}^2) G(\xi, s) = -\partial_s G(\xi, s)$ and is given by

$$G(\xi, s) = \prod_{j=1}^{d} (\cosh(a_j \xi_0 s))^{-\frac{1}{2}} e^{-\xi_j^2 s \frac{\tanh(a_j \xi_0 s)}{a_j \xi_0 s}}, \tag{2.63}$$

with the convention $b^{-1} \tanh b = 1$ for $b = 0$.

We let

$$f_{(\lambda)}(\xi) = \int_0^1 e^{(\lambda - \nu\xi_0)s} G(\xi, s) ds. \tag{2.64}$$

Then $f_{(\lambda)}$ is a Hol($\mathbb{C}$)-family of smooth functions invariant under rotations in the (ξ_j, ξ_{n+j}) planes, $j = 1, \dots, n$. Moreover we have

$$|e^{(\lambda - \nu\xi_0)s} G(\xi, s)| \leq 2^n e^{\Re\lambda - \rho(\xi)}, \qquad 0 < s \leq 1. \tag{2.65}$$

where we have set

$$\rho(\xi) = \frac{1}{2} (\sum_{j=1}^{d} |a_j| - |\Re\nu|) |\xi_0| + \sum_{j=1}^{d} \xi_j^2 \frac{\tanh(a_j \xi_0)}{a_j \xi_0}. \tag{2.66}$$

Indeed for any multi-order α there exists $C_\alpha > 0$ independent of y such that

$$|\partial_\xi^\alpha (e^{(\lambda - \nu\xi_0)s} G(\xi, s))| \leq C_\alpha (1 + \|\xi\|)^{\langle\alpha\rangle} e^{\Re\lambda - \rho(\xi)}, \qquad 0 < s \leq 1. \tag{2.67}$$

A key point in this proof is then the following lemma.

Lemma 2.27 *There exists $c = c(y) > 0$ depending continuously on y such that*

$$\rho(\xi) \geq c|\xi| \qquad \textit{for } |\xi| \geq 1. \tag{2.68}$$

Proof of the lemma. Let $|\xi| \geq 1$. Then set $\bar{\xi} = \max |\xi_j|$ and $\bar{a} = \max |a_j|$. If $|\xi_0| = \bar{\xi}$ we have

$$\rho(\xi) \geq \frac{1}{2}(\sum_{j=1}^{d} |a_j| - |\Re\nu|)|\xi_0| \geq \frac{1}{2(d+1)}(\sum_{j=1}^{d} |a_j| - |\Re\nu|)|\xi|. \tag{2.69}$$

Remember (remark 1.39) that the ellipticity condition implies $(\sum_{j=1}^{d} |a_j| - |\Re\nu|) > 0$. If $\bar{\xi} = |\xi_j|$ for some j, $1 \leq j \leq d$, then

$$\rho(\xi) \geq \bar{\xi}^2 \frac{\tanh \bar{a}\bar{\xi}}{\bar{a}\bar{\xi}} \geq \bar{\xi} \frac{1}{(d+1)\bar{a}} \tanh(\frac{\bar{a}}{d+1}), \tag{2.70}$$

for $|\xi| \geq 1$ implies $\bar{\xi} \leq (d+1)^{-1}$. Thus in both cases the inequality (2.68) holds with

$$c = c(y) = \frac{1}{d+1} \min(\sum_{j=1}^{d} |a_j| - |\Re\nu|, \frac{1}{\bar{a}} \tanh(\frac{\bar{a}}{d+1})). \tag{2.71}$$

Moreover c depends continuously on y since $\sum_{j=1}^{d} |a_j| = \text{Trace}\,|A|$ and $\bar{a}$ is equal to the spectral radius of the anti-symmetric matrix A and coincides then with its operator norm. ■

Let us go back to the proof of the proposition. The estimates (2.65) and (2.67) imply

$$|\partial_\xi^\alpha f_{(\lambda)}(\xi)| \leq C_\alpha (1 + \|\xi\|)^{\langle\alpha\rangle} (\Re\lambda - \rho(\xi))^{-1}. \tag{2.72}$$

As $-\Re\lambda \sim |\lambda|$ on any angle sector $\Theta \subset\subset \Lambda_0$ we deduce that $f_{(\lambda)}$ lies in $\text{Hol}^{-1}(\Lambda_R, C^\infty(\mathbb{R}^{d+1}))$. On the other hand we have

$$t^2 f_{(t^2\lambda)}(t.\xi) - f_{(\lambda)}(\xi) = \int_{t^2}^{1} e^{(\lambda - \nu\xi_0)s} G(\xi, s) ds, \qquad 0 < t < 1, \tag{2.73}$$

and

$$\sum_{j=1}^{d} (\xi_j^2 - \frac{1}{4} a_j^2 \xi_0^2 \partial_{\xi_j}^2) f_{(\lambda)} + (\nu\xi_0 - \lambda) f_{(\lambda)} = 1 - e^{\lambda - \nu\xi_0} G(\xi, 1). \tag{2.74}$$

Therefore (2.65) and (2.67) together with lemma 2.27 show that $f_{(\lambda)}(\xi)$ lies in $S_{-2}^{-1}(\mathbb{R}^{d+1}, \Lambda_R)$ and satisfies (2.61). Hence $f_{(\lambda)}$ is a solution of (2.42) in the normal y-coordinates

Step 3: return to the original y-coordinates and smoothness with respect to y. Let us first go back to the anti-symmetric y-coordinates. This is done by means of the change of coordinates $(x_0, x') \to (x_0, Qx')$ where Q

is the orthogonal matrix which brings the matrix A into normal form. As it is a linear change of coordinates the expression of the symbol $f_{(\lambda)}$ in the anti-symmetric y-coordinates is obtained from its expression in the normal y-coordinates by means of the change $(\xi_0, \xi') \to (\xi_0, Q^t\xi')$. So in the anti-symmetric y-coordinates it is given by

$$f_{(\lambda)}(\xi) = \int_0^1 e^{(\lambda - \nu\xi_0)s} F(\xi, s) ds. \tag{2.75}$$

By (2.49) we have

$$F(\xi, s) = G(\xi_0, Q^t\xi', s) = \det\cosh(s\xi_0 A)^{-\frac{1}{2}} e^{-s\langle \xi', (s\xi_0 A)^{-1}\tanh s\xi_0 A\rangle}. \tag{2.76}$$

In particular $f_{(\lambda)}$ depends smoothly on y, i.e. lies in $C^\infty(U \times \mathbb{R}^{d+1}) \hat{\otimes} \operatorname{Hol}(\Lambda_R)$. Moreover the estimates (2.65) and (2.67) hold uniformly with respect to y and the coefficients of the matrix Q are bounded independently of y, for Q is an orthogonal matrix. So for $0 < s < 1$ we have

$$|\partial_\xi^\alpha (e^{(\lambda - \nu\xi_0)s} F(\xi, s))| \leq C_\alpha (1 + \|\xi\|)^{\langle\alpha\rangle} e^{\Re\lambda - \rho(\xi)}, \tag{2.77}$$

with C_α independent of y.

However we can identify the space of real anti-symmetric $d \times d$ matrices with $\mathbb{R}^{\frac{d(d-1)}{2}}$. Then as functions in this space

$$\det(\cosh A)^{-1} \in M_d(\mathcal{S}(\mathbb{R}^{\frac{d(d-1)}{2}})), \qquad A^{-1}\tanh A \in M_d(C_b^\infty(\mathbb{R}^{\frac{d(d-1)}{2}})), \tag{2.78}$$

where $C_b^\infty(\mathbb{R}^{\frac{d(d-1)}{2}})$ denotes the space of smooth functions which together with all their derivatives are bounded. Therefore differentiations with respect to y harm the estimates (2.77) only by factors dominated by $C(y)(1 + \|\xi\|)^m$. Hence

$$|\partial_y^\alpha \partial_\xi^\beta (e^{(\lambda - \nu\xi_0)s} F(\xi, s))| \leq C_{\alpha\beta}(y)(1 + \|\xi\|)^{m_{\alpha\beta}} e^{\Re\lambda - \rho(\xi)}, \qquad 0 < s \leq 1. \tag{2.79}$$

for some integer $m_{\alpha\beta}$ and some locally bounded function $C_{\alpha\beta}(y)$. These estimates together with lemma 2.27 and (2.73)- (2.74) show that in the anti-symmetric coordinates $f_{(\lambda)}$ lies in $S_{-2}^{-1}(U \times \mathbb{R}^{d+1}, \Lambda_R)$ and satisfies to the equalities

$$(p_2^y - \lambda) *^y f_{(\lambda)}^y = 1 = f_{(\lambda)}^y *^y (p_2^y - \lambda) \qquad \bmod S^{-\infty, -\infty}(U \times \mathbb{R}^{d+1}, \Lambda_R). \tag{2.80}$$

Let us finally return to the original y-coordinates. The change of coordinates involved is

$$\phi^y(x) = (x_0 + \frac{1}{4}\sum_{j,k=1}^d (c_{jk} + c_{kj}) x_j x_k, x'). \tag{2.81}$$

As $\phi_y(x)$ is an isomorphism from the anti-symmetric y-group onto the original y-group, $f_{(\lambda)}$ transforms into

$$(\phi_{y*}\check{f}_{(\lambda)})^\wedge = (\check{f}_{(\lambda)} \circ \phi_y^{-1})^\wedge. \tag{2.82}$$

Since ϕ_y depends smoothly on y and is homogenous of degree 1 with respect to the Heisenberg dilations, it maps $S^{-1}_{-2}(U \times \mathbb{R}^{d+1}, \Lambda_R)$ and $S^{-\infty,-\infty}(U \times \mathbb{R}^{d+1}, \Lambda_R)$ into themselves. Hence in the original y-coordinates $f_{(\lambda)}$ lies in $S^{-1}_{-2}(U \times \mathbb{R}^{d+1}, \Lambda_R)$ and satisfies

$$(p_2^y - \lambda) *^y f^y_{(\lambda)} = 1 = f^y_{(\lambda)} *^y (p_2^y - \lambda) \qquad \bmod S^{-\infty,-\infty}(U \times \mathbb{R}^{d+1}, \Lambda_R), \tag{2.83}$$

which means that $f_{(\lambda)}(y,\xi) = f^y_{(\lambda)}(\xi)$ inverts $p_2(y,\xi) - \lambda$ modulo $S^{-\infty,-\infty}(U \times \mathbb{R}^{d+1}, \Lambda_R)$. ■

Proposition 2.28 *Let $(M, \mathcal{V})$ be a Heisenberg manifold and Δ an elliptic sublaplacian on M. Then for any $R > 0$ there exists $Q_{(\lambda)} \in \Psi_{\mathcal{V}}^{-1,-2}(M, \Lambda_R)$ such that*

$$(\Delta - \lambda) Q_{(\lambda)} = 1 = Q_{(\lambda)}(\Delta - \lambda) \qquad \bmod \Psi_{\mathcal{V}}^{-1,-\infty}(M, \Lambda_R). \tag{2.84}$$

Proof. Let us first work on an open subset of $\mathbb{R}^{d+1}$ on which we have

$$\Delta = -\sum_{j=1}^{d} X_j^2 + i\nu(x) X_0 + \text{ lower terms}, \tag{2.85}$$

where $X_0, X_1, \ldots, X_d$ is a $\mathcal{V}$-frame for TU. Let $p_2(x,\xi)$ be the principal symbol of Δ. By proposition 2.26 there exists $f_{(\lambda),-2}$ in $S^{-1}_{-2}(U \times \mathbb{R}^{d+1}, \Lambda_R)$ such that

$$(p_2 - \lambda) * f_{(\lambda),-2} = 1 = f_{(\lambda),-2} * (p_2 - \lambda) \qquad \bmod S^{-\infty,-\infty}(U \times \mathbb{R}^{d+1}, \Lambda_R). \tag{2.86}$$

Let $Q_{(\lambda)} \in \Psi_{\mathcal{V}}^{-1,-2}(U, \Lambda_R)$ be uniformly properly supported with symbol $f_{(\lambda)}$. By proposition 2.14 the symbol of $\Delta Q_{(\lambda)}$ lies in $S^{0,-1}(U \times \mathbb{R}^{d+1}, \Lambda_R)$ and its principal symbol is equal to

$$p_2 * f_{(\lambda),-2} = (p_2 - \lambda) * f_{(\lambda),-2} + \lambda f_{(\lambda),-2} = 1 + \lambda f_{(\lambda),-2}. \tag{2.87}$$

As $\lambda f_{(\lambda)}$ is the symbol of $\lambda Q_{(\lambda)}$ it follows that

$$(\Delta - \lambda) Q_{(\lambda)} = 1 - R_{(\lambda)}. \tag{2.88}$$

with $R_{(\lambda)} \in \Psi_{\mathcal{V}}^{-1,-1}(U, \Lambda_R)$ uniformly properly supported.

For $k \geq 0$ denote by $r^{(k)}_{(\lambda)}$ the symbol of $R^k_{(\lambda)}$. It follows from proposition 2.8 that there exists $r_{(\lambda)}$ in $S^{0,-1}(U \times \mathbb{R}^{d+1}, \Lambda_R)$ such that $r_{(\lambda)} \sim \sum_{k \leq 0} r^{(k)}_{(\lambda)}$. Then we have

$$(1 - R_{(\lambda)}) r_{(\lambda)}(x, \sigma(x, D)) = 1 \qquad \bmod \Psi_{\mathcal{V}}^{-1,-\infty}(U, \Lambda_R). \tag{2.89}$$

So setting $Q'_{(\lambda)} = Q_{(\lambda)} r_{(\lambda)}(x, \sigma(x, D))$ we get

$$(\Delta - \lambda) Q'_{(\lambda)} = 1 \qquad \bmod \Psi_{\mathcal{V}}^{-1,-\infty}(U, \Lambda_R). \tag{2.90}$$

Now, let (φ_i) be a locally finite partition of unity subordinated to a locally finite open cover (U_i) on which Δ takes the form (2.85). Then for each index i there exists $Q_{i(\lambda)} \in \Psi_{\mathcal{V}}^{-1,-2}(U_i, \Lambda_R)$ such that

$$(\Delta - \lambda) Q_{i(\lambda)} = 1 \qquad \bmod \Psi_{\mathcal{V}}^{-1,-\infty}(U_i, \Lambda_R). \tag{2.91}$$

Pick then some $\psi_i \in C_c^\infty(U_i)$ such that $\psi_i = 1$ near $\operatorname{supp} \varphi_i$ and set

$$Q_{(\lambda)} = \sum \psi_i Q_{i(\lambda)} \varphi_i \in \Psi_{\mathcal{V}}^{-1,-2}(M, \Lambda_R), \tag{2.92}$$

Then we have

$$\begin{aligned} (\Delta - \lambda) Q_{(\lambda)} &= \sum (\Delta - \lambda) \psi_i Q_{i(\lambda)} \varphi_i, \\ &= \sum [\Delta, \psi_i] Q_{i(\lambda)} \varphi_i + \sum \psi_i (\Delta - \lambda) Q_{i(\lambda)} \varphi_i, \\ &= 1 \qquad \bmod \Psi_{\mathcal{V}}^{-1,-\infty}(M, \Lambda_R). \end{aligned} \tag{2.93}$$

Similarly we can construct $Q'_{(\lambda)} \in \Psi_{\mathcal{V}}^{-1,-2}(M, \Lambda_R)$ such that

$$Q'_{(\lambda)}(\Delta - \lambda) = 1 \qquad \bmod \Psi_{\mathcal{V}}^{-1,-\infty}(M, \Lambda_R). \tag{2.94}$$

Necessarily $Q_{(\lambda)} - Q'_{(\lambda)} \in \Psi_{\mathcal{V}}^{-1,-\infty}(M, \Lambda_R)$, so $Q_{(\lambda)}$ is a two-side parametrix for $\Delta - \lambda$ modulo $\Psi_{\mathcal{V}}^{-1,-\infty}(M, \Lambda_R)$ and the proof is complete. ■

Remark 2.29 The parametrix constructed above is a parametrix modulo $\Psi_{\mathcal{V}}^{-1,-\infty}(M, \Lambda_R)$ and not modulo $\Psi_{\mathcal{V}}^{-\infty,0}(M, \Lambda_R)$, as we could have expected since $(\Delta - \lambda)$ lies in $\Psi_{\mathcal{V}}^{1,2}(M, \Lambda_R)$ and $Q_{(\lambda)}$ in $\Psi_{\mathcal{V}}^{-1,2}(M, \Lambda_R)$. This is an essential point for showing the existence of rays of minimal growth for Δ. In fact if we let $R_{(\lambda)} = 1 - (\Delta - \lambda) Q_{(\lambda)}$ and we replace $Q_{(\lambda)}$ by

$$Q_{(\lambda)}(1 + R_{(\lambda)} + \ldots + R^{p-1}_{(\lambda)}), \tag{2.95}$$

then for any integer p we get a parametrix modulo $\Psi_{\mathcal{V}}^{-p,-\infty}(M, \Lambda_R)$.

2.7 Rays of minimal growth for sublaplacians

In this section we show the existence of rays of minimal growth for an elliptic sublaplacian Δ on a compact Heisenberg manifold $(M, \mathcal{V})$.

Definition 2.30 *A ray $L \subset \mathbb{C}$ is a ray of minimal growth for Δ if $\Delta - \lambda$ is invertible for any $\lambda \in L$ and the norm of the resolvent $\|(\Delta - \lambda)^{-1}\|$ is* $\mathrm{O}(1/|\lambda|)$ *on L.*

For $r > 0$ and Θ an open angle sector we set

$$\Theta_r = \{\lambda \in \Theta; |\lambda| > r\}. \tag{2.96}$$

Theorem 2.31 *On any angle sector $\Theta \subset\subset \Lambda_0$ there only finitely many eigenvalues for Θ moreover on $\Theta \setminus \operatorname{sp} \Delta$ we have*

$$\|(\Delta - \lambda)^{-1}\| \leq C_\Theta |\lambda|^{-1}. \tag{2.97}$$

Therefore each ray contained in Θ, except maybe a finite number, is a ray of minimal growth for Δ.

Proof. By proposition 2.28 there exists $Q(\lambda) \in \Psi_{\mathcal{V}}^{-1,-2}(U, \Lambda_0)$ such that

$$R_{(\lambda)} = 1 - Q_{(\lambda)}(\Delta - \lambda), \qquad \lambda \in \Lambda_0, \tag{2.98}$$

is a $\mathrm{Hol}^{-1}(\Lambda_1)$-family of smoothing operators. As we have $\Theta \subset\subset \Lambda_0$ proposition 2.25 implies

$$\|R_{(\lambda)}\| \leq C_\Theta (1 + |\lambda|)^{-1}, \qquad \lambda \in \Theta. \tag{2.99}$$

Thus there exists $r > 0$ such that $\|R_{(\lambda)}\| \leq \frac{1}{2}$ on $\Theta_r = \{\lambda \in \Theta; |\lambda| > r\}$. Then $1 - R_{(\lambda)}$ is invertible on $L^2(M)$ and $\|(1 - R_{(\lambda)})^{-1}\| \leq 2$. Hence

$$(1 - R_{(\lambda)})^{-1} Q_{(\lambda)}(\Delta - \lambda) = (1 - R_{(\lambda)})^{-1}(1 - R_{(\lambda)}) = 1, \qquad \lambda \in \Theta_r. \tag{2.100}$$

However by (2.25) the family $Q_{(\lambda)}$ lies in $\mathrm{Hol}^{-1}(\Lambda_1, \mathcal{L}(L^2(M))$. Thus for any $\lambda \in \Theta_r$ the operator $(1 - R_{(\lambda)})^{-1} Q_{(\lambda)}$ is a left inverse of $\Delta - \lambda$ on $L^2(M)$ and

$$\|(1 - R_{(\lambda)})^{-1} Q_{(\lambda)}\| \leq C_{\Theta r} |\lambda|^{-1}, \qquad \lambda \in \Theta_r. \tag{2.101}$$

Similarly enlarging r if necessary we can construct a right inverse on $L^2(M)$ satisfying the above estimates. Then $\Delta - \lambda$ is invertible on $L^2(M)$ for any λ in Θ_r with a resolvent satisfying to (2.97). ■

Corollary 2.32 *Any selfadjoint elliptic sublaplacian on a compact Heisenberg manifold is bounded from below and thus has a heat kernel asymptotic as in [BGS] and proposition 1.53.*

We can now prove the main result of this chapter.

Theorem 2.33 *Let $(M, \mathcal{V})$ a compact Heisenberg manifold and Δ an elliptic sublaplacian Δ on M. Then there exist $R > 0$ and an open pseudocone Λ containing $D(0, R) \setminus 0$, contained in*

$$\Lambda_R = \{\lambda \in \mathbb{C}\backslash 0;\ \Re\lambda < 0 \text{ or } |\lambda| < R\}, \tag{2.102}$$

such that:

- *(i) for any $\lambda \in \Lambda$ the operator $\Delta - \lambda$ is invertible on $L^2(M)$;*
- *(ii) the family $(\Delta - \lambda)^{-1}$, $\lambda \in \Lambda$, lies in $\Psi_{\mathcal{V}}^{-1,-2}(M, \Lambda)$;*
- *(iii) for any pseudocone $\Lambda' \subset\subset \Lambda$,*

$$\|(\Delta - \lambda)^{-1}\| \leq C_{\Lambda'}(1 + |\lambda|)^{-1}, \qquad \lambda \in \Lambda'. \tag{2.103}$$

In particular each ray contained in Λ is a ray of minimal growth for Δ.

Proof. Consider an open angle sector $\Theta \subset\subset \Lambda_0$. By proposition 2.31 there exists $r > 0$ such that $\operatorname{sp}\Delta \cap \Theta_r = \emptyset$. As $\operatorname{sp}\Delta$ is discrete $\operatorname{sp}\Delta \cap \bar{D}(0, r)$ is finite and there are only finitely many rays $L_1, \ldots, L_k$ contained in Θ and intersecting $\operatorname{sp}\Delta$.

On the other hand as 0 is at most an isolated point in the spectrum of Δ there exists $R > 0$ such that $\operatorname{sp}\Delta \cap D(0, R) \subset \{0\}$. Therefore the set

$$\Lambda = (D(0, R) \setminus \{0\}) \cup (\Theta \setminus (L_1 \cup \ldots \cup L_k)). \tag{2.104}$$

is an open pseudocone contained in Λ_R, and for any $\lambda \in \Lambda$ the operator $\Delta - \lambda$ is invertible on $L^2(M)$.

However the proposition 2.28 yields $Q_{(\lambda)}$ in $\Psi_{\mathcal{V}}^{-1,-2}(U, \Lambda_R)$ and $R_{(\lambda)}$ in $\Psi_{\mathcal{V}}^{-1,-\infty}(U, \Lambda_R)$ such that

$$Q_{(\lambda)}(\Delta - \lambda) = 1 - R_{(\lambda)}. \tag{2.105}$$

Then we have

$$(\Delta - \lambda)^{-1} = Q_{(\lambda)} + R_{(\lambda)}(\Delta - \lambda)^{-1}. \tag{2.106}$$

Since $(\Delta - \lambda)^{-1}$ is an analytic family of bounded operators, proposition 2.31 the family $(\Delta - \lambda)^{-1}$ shows that it is a $\mathrm{Hol}^{-1}(\Lambda)$-family with values in $\mathcal{L}(L^2(M))$. Therefore $R_{(\lambda)}(\Delta - \lambda)^{-1}$ is a $\mathrm{Hol}^{-1}(\Lambda)$-family of smoothing operators, which implies that $(\Delta - \lambda)^{-1}$ lies in $\Psi_{\mathcal{V}}^{-1,-2}(U, \Lambda_1)$. Then (2.103) is a consequence of proposition 2.25. ■

Chapter 3

Holomorphic families of $\Psi_{\mathcal{V}}DO$ operators and complex powers of sublaplacians

The aim of this chapter is to introduce holomorphic families of $\Psi_{\mathcal{V}}DO$ operators and to construct the complex powers of an elliptic sublaplacian as a holomorphic 1-parameter group of $\Psi_{\mathcal{V}}DO$'s. In section 3.1 we define holomorphic families of $\Psi_{\mathcal{V}}DO$ operators and study their main properties. In section 3.2 we can study in terms of holomorphic families the complex powers of a non-negative elliptic sublaplacian with the help of the pseudodifferential construction of heat-kernels given in [BGS] (theorem 3.17). Finally in section 3.3 we follow [Se] to define the complex powers of an elliptic sublaplacian by means of the parametric $\Psi_{\mathcal{V}}DO$ calculus developed in chapter 2 (theorems 3.21 and 3.22).

3.1 Holomorphic families of $\Psi_{\mathcal{V}}DO$ operators

In all this section Ω is an open domain in $\mathbb{C}$ and U is an open subset of $\mathbb{R}^{d+1}$ equipped with a hyperplane bundle $\mathcal{V} \subset TU$ and $\mathcal{V}$-frame $X_0, X_1, \ldots, X_d$ of TU. Then $\sigma(x,\xi)$ and ε_x denotes the (real) symbol of the $\mathcal{V}$-frame and the affine change onto the x-coordinates.

Definition 3.1 *A family $(f_z) \subset S^*(U \times \mathbb{R}^{d+1})$ indexed by Ω is holomorphic if the following conditions hold:*

(i) the order m_z of the symbol f_z depends holomorphically on z;

(ii) for $(x,\xi) \in U \times \mathbb{R}^{d+1}$ fixed the function $z \to f_z(x,\xi)$ is holomorphic on Ω;

(iii) the bounds of the asymptotic expansion

$$f_z(x,\xi) \sim \sum_{j\geq 0} f_{z,m_z-j}(x,\xi), \qquad f_{z,l} \in S_l(U\times\mathbb{R}^{d+1}), \tag{3.1}$$

are locally uniform with respect to z.

The space of holomorphic families of symbols is denoted $\mathrm{Hol}(\Omega, S^*(U\times\mathbb{R}^{d+1})$.

Remark 3.2 The axiom (iii) requires that for any integer N we have

$$|\partial_x^\alpha \partial_\xi^\beta (f_z - \sum_{j<N} f_{z,m_z-j})(x,\xi)| \leq C_{\alpha\beta N}(x,z)\|\xi\|^{\Re m_z - N - \langle\beta\rangle}, \qquad \|\xi\|\geq 1, \tag{3.2}$$

where $C_{\alpha\beta n}(x,z)$ is a locally bounded function on $U\times\Omega$. By [Ho1, theorem 2.9] this equivalent to the condition: for any $\Omega' \subset\subset \Omega$ and any integer N if J large enough we have

$$|\partial_x^\alpha \partial_\xi^\beta (f_z - \sum_{j\leq J} f_{z,m_z-j})(x,\xi)| \leq C_{\alpha\beta N J\Omega'}(x)|\xi|^{-N}, \qquad |\xi|\geq 1, \tag{3.3}$$

with $C_{\alpha\beta nj\Omega'}(x)$ locally bounded function on U.

Remark 3.3 It follows from the above axioms that the homogeneous symbols $f_{j,z}(x,\xi)$ depends holomorphically on z. Indeed for the principal symbol we have pointwise

$$f_{0,z}(x,\xi) = \lim_{\lambda\to+\infty} \lambda^{-m_z} f_z(x,\lambda.\xi), \qquad \xi\neq 0. \tag{3.4}$$

By the axioms (i) and (ii) the right-hand side is holomorphic in z and by the last axiom this family is bounded in $\mathrm{Hol}(\Omega, C^\infty(U\times(\mathbb{R}^{d+1}\backslash 0))$. Hence the above limit converges in $\mathrm{Hol}(\Omega, C^\infty(U\times(\mathbb{R}^{d+1}\backslash 0))$ and $f_{0,z}$ depends holomorphically on z. Similarly for any $j>0$ we have

$$f_{j,z}(x,\xi) = \lim_{\lambda\to+\infty} \lambda^{j-m_z}(f_z(x,\lambda.\xi) - \sum_{l<j} \lambda^{m_z-l} f_{l,z}(x,\xi)), \qquad \xi\neq 0. \tag{3.5}$$

So by induction we can show that all the symbols $f_{j,z}$ are holomorphic with respect to z.

Definition 3.4 *A family $(f_z)\subset S^*_{ah}(U\times\mathbb{R}^{d+1})$ over Ω is holomorphic if it fulfills the following conditions:*

(i) the order m_z of the symbol f_z depends holomorphically on z;

(ii) for $(x,\xi)\in U\times\mathbb{R}^{d+1}$ fixed the function $z\to f_z(x,\xi)$ is holomorphic on Ω;

(iii) for any $t > 0$ the family

$$f_z(x, t.\xi) - t^{m_z} f(x, \xi), \qquad z \in \Omega, \tag{3.6}$$

is a holomorphic family with values in $S^{-\infty}(U \times \mathbb{R}^{d+1})$.

The space of holomorphic S^*_{ah}-valued families is denoted $\mathrm{Hol}(\Omega, S^*_{ah}(U \times \mathbb{R}^{d+1})$.

Lemma 3.5 *A holomorphic family of almost homogeneous symbols is a holomorphic family of symbols.*

Proof. Let (f_z) be a holomorphic S^*_{ah}-valued family and let us show it is a holomorphic family of symbols. As the first two conditions of definition 3.1 are fulfilled we need only to check the last one. In fact it follows from lemma 1.10 and the property (iii) of definition 3.4 there exists a family $(g_z) \subset C^\infty(U \times (\mathbb{R}^{d+1} \backslash 0))$ such that g_z is homogeneous of degree m_z and for any integer N we have

$$|\partial_x^\alpha \partial_\xi^\beta (f_z(x, \xi) - g_z(x, \xi))| \leq C_{\alpha\beta N}(x, z) \|\xi\|^{-N}, \qquad \xi \neq 0, \tag{3.7}$$

with $C_{\alpha\beta N}(x, z)$ locally bounded on $U \times \Omega$. This means that (f_z) satisfies the last condition and completes the proof. ■

Therefore we have the following characterization of holomorphic families of symbols.

Proposition 3.6 *A family (f_z) of symbols over Ω with order m_z is a holomorphic family of symbols if, and only if, it fulfills the following properties:*

(i) the order m_z depends holomorphically on z;

(ii) for $(x, \xi) \in U \times \mathbb{R}^{d+1}$ fixed the function $z \to f_z(x, \xi)$ is holomorphic on Ω;

(iii) there exist a holomorphic families $(f_{j,z})$, $j = 0, 1, \ldots$, of almost homogeneous symbols with $\mathrm{ord} f_{j,z} = m_z - j$ *such that we have an asymptotic expansion $f_z \sim \sum f_{j,z}$ with locally uniform bounds in z.*

Definition 3.7 *A family $(P_z) \subset \Psi^*_{\mathcal{V}}(U)$ is holomorphic if, and only if, P_z is in the form*

$$P_z = f_z(x, \sigma(x, D)) + R_z, \tag{3.8}$$

with (f_z) and (R_z) holomorphic families of symbols and smoothing operators. The space of holomorphic families of $\Psi_{\mathcal{V}}DO$ operators is denoted $\mathrm{Hol}(\Omega, \Psi^*_{\mathcal{V}}(U))$.

Proposition 3.8 *Let (P_z) be a holomorphic family of $\Psi_{\mathcal{V}}DO$'s over Ω. Then:*

1) (P_z) defines a holomorphic family with values in $\mathcal{L}(C_c^\infty(U), C^\infty(U))$ and $\mathcal{L}(\mathcal{E}'(U)), \mathcal{D}'(U))$.

2) The kernel of P_z is given outside the diagonal of $U \times U$ by a holomorphic family of smooth functions.

3) We can write P_z as $P_z = Q_z + R_z$, where (Q_z) is a holomorphic family of uniformly properly supported $\Psi_{\mathcal{V}}DO$'s and (R_z) is a holomorphic family of smoothing operators.

4) If the family (P_z) is uniformly properly supported, it gives rise to holomorphic families of continuous operators from $C_c^\infty(U)$, $C^\infty(U)$, $\mathcal{E}'(U)$ and $\mathcal{D}'(U)$ to themselves.

Proof. To prove 1) and 2) it is enough to consider the case $P_z = f_z(x, \sigma(x, D))$ with (f_z) holomorphic family of symbols. Shrinking Ω if necessary we can suppose the order m_z of f_z stays bounded. Then there exists a real k such that the family (f_z) is holomorphic with values in $S^k_{||}(U \times \mathbb{R}^{d+1})$.

As noted in the proof of proposition 2.11 the map $f \to f(x, \sigma(x, D))$ is continuous from $S^k_{||}(U \times \mathbb{R}^{d+1})$ into $\mathcal{L}(\mathcal{E}'(U)), \mathcal{D}'(U))$. So by the closed graph theorem it is also continuous from $S^k_{||}(U \times \mathbb{R}^{d+1})$ into $\mathcal{L}(C_c^\infty(U), C^\infty(U))$. As these maps are $\mathbb{C}$-linear it follows that the family of operators $f_z(x, \sigma(x, D))$ is holomorphic with values in $\mathcal{L}(C_c^\infty(U), C^\infty(U))$ and $\mathcal{L}(\mathcal{E}'(U)), \mathcal{D}'(U))$.

Similarly the map which assigns to the symbol $f \in S^k_{||}(U \times \mathbb{R}^{d+1})$ the restriction of the kernel of $f(x, \sigma(x, D))$ to $U \times U \setminus \Delta$ is a continuous $\mathbb{C}$-linear map from $S^k_{||}(U \times \mathbb{R}^{d+1})$ into $C^\infty(U \times U \setminus \Delta)$. So the kernel of $f_z(x, \sigma(x, D))$ is given outside Δ by a holomorphic family of smooth functions.

Finally the proofs of the assertions 3) and 4) follow along the same lines of the proof of proposition 2.13. ■

Proposition 3.9 *Let $(P_{1,z})$ and $(P_{2,z})$ be two holomorphic families of $\Psi_{\mathcal{V}}DO$ operators, one of them being uniformly properly supported. Then the family $P_z = P_{1,z}P_{2,z}$ is a holomorphic family of $\Psi_{\mathcal{V}}DO$ operators over Ω.*

Proof. By definition of a holomorphic family of $\Psi_{\mathcal{V}}DO$'s we have

$$(3.9)\qquad P_{1,z} = f_{1,z}(x, \sigma(x, D)) + R_{1,z}, \qquad P_{2,z} = f_{2,z}(x, \sigma(x, D)) + R_{2,z},$$

with $(f_{1,z})$, $(f_{2,z})$ holomorphic families of symbols and $(R_{1,z})$, $(R_{2,z})$ holomorphic families of smoothing operators. So by proposition 3.8 up to a holomorphic family of smoothing operators we have

$$(3.10)\qquad P_z = P_{1,z}P_{2,z} = \sum \varphi_i (f_{1,z} \#_{\psi_i} f_{2,z})(x, \sigma(x, D)),$$

where $(\varphi_i) \subset C_c^\infty(U)$ is a locally finite partition of unity and $(\psi_i) \subset C_c^\infty(U)$ is such that $\psi_i = 1$ near $\operatorname{supp} \varphi_i$. Therefore it is enough to show that given $\psi \in C_c^\infty(U)$ the family

$$f_{1,z} \#_\psi f_{2,z}, \qquad z \in \Omega, \tag{3.11}$$

is a holomorphic family of symbols.

However shrinking Ω if necessary, we can suppose the orders of $f_{1,z}$ and $f_{2,z}$ stay bounded. Thereby the families $f_{1,z}$ and $f_{2,z}$ are holomorphic with values in $S_{||}^k(U \times \mathbb{R}^{d+1})$ for some real k. Then it follows from the lemma 1.19 and the remark 1.22 that the axioms (ii) and (iii) of the definition 3.1 are satisfied by the family $f_{1,z} \#_\psi f_{2,z}$. As the first axiom is obviously satisfied $f_{1,z} \#_\psi f_{2,z}$ is a holomorphic family with values in $S^*(U \times \mathbb{R}^{d+1})$ and the proof is achieved. ∎

Let us now define holomorphic families of kernels and obtain a characterization of holomorphic families of $\Psi_{\mathcal{V}}DO$ operators.

There is a technical difficulty for defining holomorphic families with values in $\mathcal{K}_*(U \times \mathbb{R}^{d+1})$ since when the order crosses positive integers the homogeneity of the distributions breaks down and logarithmic terms appear. As with $\Psi_{\mathcal{V}}DO$'s with parameter this can be avoided if we consider instead holomorphic families of almost homogenous distributions.

Definition 3.10 *A family $(K_z) \subset \mathcal{K}_{ah}^*(U \times \mathbb{R}^{d+1})$ over Ω is holomorphic if it satisfies the following properties:*

(i) the order m_z of K_z is a holomorphic function on Ω;

(ii) the family (K_z) is holomorphic with values in $C^\infty(U)\hat{\otimes}\mathcal{D}'(\mathbb{R}^{d+1})$ and K_z is given on $U \times (\mathbb{R}^{d+1} \backslash 0)$ by a holomorphic family of smooth functions;

(iii) for any $t > 0$ the family

$$K_z(x, t.y) - t^{m_z} K_z(x, y), \qquad z \in \Omega, \tag{3.12}$$

is holomorphic with values in $C^\infty(U \times \mathbb{R}^{d+1})$.

The space of holomorphic $\mathcal{K}_{\mathrm{ah}}^*(U \times \mathbb{R}^{d+1})$-valued families is denoted $\operatorname{Hol}(\Omega, \mathcal{K}_{ah}^*(U \times \mathbb{R}^{d+1}))$.

Definition 3.11 *A family $(K_z) \subset \mathcal{K}^*(U \times \mathbb{R}^{d+1})$ over Ω is holomorphic if the following conditions hold:*

(i) the order m_z of K_z depends holomorphically on z;

(ii) there exist holomorphic families of almost homogeneous kernels $(K_{j,z})$, $j = 0, 1, \ldots$, with $\operatorname{ord} K_{j,z} = m_z + j$ *such that we have an asymptotic expansion*

$$K_z \sim \sum K_{j,z}, \tag{3.13}$$

in the sense that for any open $\Omega' \subset\subset \Omega$ and any integer N if J is large enough we have

$$K_z - \sum_{j \leq J} K_{z,m_z+j} \in \operatorname{Hol}(\Omega', C^N(U \times \mathbb{R}^{d+1})). \tag{3.14}$$

The space of holomorphic $\mathcal{K}^*(U \times \mathbb{R}^{d+1})$-valued families is denoted $\operatorname{Hol}(\Omega, \mathcal{K}^*(U \times \mathbb{R}^{d+1}))$.

Remark 3.12 This definition implies that such a family is actually a holomorphic family with values in $C^\infty(U)\hat{\otimes}\mathcal{D}'(\mathbb{R}^{d+1})$ and it is given on $U \times (\mathbb{R}^{d+1} \backslash 0)$ by a holomorphic family of smooth functions.

This leads to the following characterization of holomorphic families of $\Psi_{\mathcal{V}}DO$ operators:

Proposition 3.13 *Let (P_z) be a family of $\Psi_{\mathcal{V}}DO$'s over Ω. Then it is a holomorphic family of $\Psi_{\mathcal{V}}DO$'s if, and only if, the kernel of P_z is in the form*

$$k_z(x, y) = K_z(x, \varepsilon_x(y)) + R_z(x, y), \tag{3.15}$$

with (K_z) and (R_z) holomorphic families with values in $\mathcal{K}^*(U \times \mathbb{R}^{d+1})$ and $C^\infty(U \times U)$.

Proof. The arguments of the proof of proposition 1.29 apply to holomorphic families if, as in the proof of proposition 2.19, we replace the homogeneity of symbols and kernels by almost homogeneity. ■

Using the above kernel characterization we can obtain the invariance by Heisenberg diffeomorphisms of holomorphic families of $\Psi_{\mathcal{V}}DO$ operators.

Proposition 3.14 *Let $\phi : U \to \tilde{U}$ be a Heisenberg diffeomorphism from U onto $\tilde{U}$ where $\tilde{U}$ is an open subset of $\mathbb{R}^{d+1}$ equipped with a hyperplane bundle $\tilde{\mathcal{V}} \subset T\tilde{U}$ and a $\tilde{\mathcal{V}}$-frame. For any holomorphic family $\tilde{P}_z$ of $\Psi_{\tilde{\mathcal{V}}}DO$ operators on $\tilde{U}$ the family $P_z = \phi^*\tilde{P}_z$ is a holomorphic family of $\Psi_{\mathcal{V}}DO$ operators on U.*

Proof. Let (K_z) be a holomorphic family with values in $\mathcal{K}^*(U \times \mathbb{R}^{d+1})$ over Ω. By shrinking Ω if necessary we can suppose that the order of K_z stays bounded. Then it follows from lemma 2.20 and proposition 3.13 there exists a real $k \geq 0$ such that for $\langle\beta\rangle$ large enough we have

$$|\partial_x^\alpha \partial_y^\beta K_z(x,y)| \leq C_{k\alpha\beta}(x,z)\|y\|^{-k-\langle\beta\rangle}, \qquad 0 < \|y\| \leq 1, \tag{3.16}$$

with $C_{k\alpha\beta}(x,z)$ locally bounded on $U \times \Omega$. Together with proposition 3.13 these estimates allow the arguments of the proof of proposition 1.33 to work *verbatim* with holomorphic families of $\Psi_{\mathcal{V}}DO$ operators, thereby proving the invariance by Heisenberg diffeomorphism. ■

We can now define holomorphic families of $\Psi_{\mathcal{V}}DO$'s on any Heisenberg manifold.

Definition 3.15 *Let $(M, \mathcal{V})$ be a Heisenberg manifold. A family $(P_z) \subset \Psi^*_{\mathcal{V}}(M)$ over Ω is a holomorphic it satisfies to the following:*

(i) the order m_z of P_z depends holomorphically on z;

(ii) for φ, ψ in $C^\infty(M)$ with disjoint supports $\varphi P_z \psi$ is given by a holomorphic family of smooth kernels.

(iii) On any local Heisenberg chart P_z is given by a holomorphic family of $\Psi_{\mathcal{V}}DO$'s on an open subset of $\mathbb{R}^{d+1}$ equipped with a $\mathcal{V}$-frame.

All the preceding properties of holomorphic families of $\Psi_{\mathcal{V}}DO$'s on an open subset of $\mathbb{R}^{d+1}$ continue to hold in the case of manifolds. In a case of a compact manifold we have:

Proposition 3.16 *Let $(M, \mathcal{V})$ be a compact Heisenberg manifold. Let $P_{1,z}$ and $P_{2,z}$ be two holomorphic families of $\Psi_{\mathcal{V}}DO$'s on M. Then $P_z = P_{1,z}P_{2,z}$ is also a holomorphic family of $\Psi_{\mathcal{V}}DO$ operators.*

3.2 Complex powers of an elliptic sublaplacian (positive case)

Let Δ be a formally selfdajoint elliptic sublaplacian on a compact Heisenberg manifold $(M^{d+1}, \mathcal{V})$. We assume here that we have $\Delta \geq c > 0$ with respect to the inner product induced by a smooth non-negative density on M.

In this section we shall study the complex powers Δ^s, $s \in \mathbb{C}$, of Δ from the point of view of holomorphic families of $\Psi_{\mathcal{V}}DO$ operators. These operators are well defined by functional calculus as unbounded operators on $L^2(M)$ and they give rise to a 1-parameter group.

Theorem 3.17 *The family $(\Delta^s)_{s\in\mathbb{C}}$ of the complex powers of Δ is a holomorphic family of $\Psi_{\mathcal{V}}DO$ operators.*

Proof. As Δ^s is a 1-parameter group and the product of $\Psi_{\mathcal{V}}DO$ operators is holomorphic by proposition 3.16, we need only to show that the family (Δ^s) is a holomorphic family of $\Psi_{\mathcal{V}}DO$'s over the left half-plane $\{\Re s < 0\}$. By Mellin formula for $\Re s > 0$ we have

$$\Delta^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-t\Delta} \frac{dt}{t}. \tag{3.17}$$

As $\int_1^\infty t^s e^{-t\Delta} \frac{dt}{t} = e^{-\frac{1}{2}\Delta}(\int_0^\infty t^s e^{-t\Delta}\frac{dt}{t})e^{-\frac{1}{2}\Delta}$ is a holomorphic family of smoothing operators, it is enough to check that

$$D_s = \int_0^1 t^s e^{-t\Delta}\frac{dt}{t}, \qquad \Re s > 0. \tag{3.18}$$

is a holomorphic family of $\Psi_{\mathcal{V}}DO$ operators.

However by proposition 1.52 the operator $\Delta + \frac{\partial}{\partial t}$ is invertible on $C_c^\infty(M\times\mathbb{R})$ and its inverse $Q = (\Delta + \frac{\partial}{\partial t})^{-1}$ belongs to $\Psi_{\mathcal{V},h}^{-2}(U\times\mathbb{R})$. If $k_Q(x,y,u-t)$ denotes its kernel then $k_Q(x,y,t)$ is the kernel of $e^{t\Delta}$ for $t>0$. Thus the kernel of D_s is

$$k_s(x,y) = \int_0^1 t^s k_Q(x,y,t)\frac{dt}{t}, \qquad \Re s > 0. \tag{3.19}$$

For φ and ψ in $C^\infty(M)$ with disjoint supports the operator $\varphi Q \psi$ is smoothing, so that $\varphi(x)k_Q(x,y,t)\psi(y)$ is smooth and $\varphi(x)k_s(x,y)\psi(y)$ is a holomorphic family of smooth kernels.

Moreover we locally have

$$k_Q(x,y,t) = |\varepsilon_x'(y)| K(x,\varepsilon_x(y),t) + R(x,y,t), \tag{3.20}$$

where K is in $\mathcal{K}_h^{-(d+2)}(U\times\mathbb{R}^{d+1}\times\mathbb{R})$, the kernel R is smooth and U is an open subset of $\mathbb{R}^{d+1}$ together with a $\mathcal{V}$-frame. As $\int_0^1 t^s R(x,y,t)\frac{dt}{t}$ is a holomorphic family of smooth functions, it remains to prove that

$$K_s(x,y) = \int_0^1 t^s K(x,y,t)\frac{dt}{t}, \qquad \Re s > 0, \tag{3.21}$$

is a holomorphic family with values in $\mathcal{K}^*(U\times\mathbb{R}^{d+1})$.

Consider now the asymptotic expansion for K in $\mathcal{K}_h^*(U\times\mathbb{R}^{d+1}\times\mathbb{R})$, i.e.

$$K(x,y,t) \sim \sum K_{j-(d+2)}(x,y,t), \qquad K_l \in \mathcal{K}_{h,l}(U\times\mathbb{R}^{d+1}\times\mathbb{R}), \tag{3.22}$$

Since for $j = 0, 1, \dots$ the kernel $K_{j-(d+2)}$ has integrable order on $\mathbb{R}^{d+1} \times \mathbb{R}$, hence lies in $C^\infty(U) \otimes L^1_{\text{loc}}(\mathbb{R}^{d+2})$, we define holomorphic families with values in $C^\infty(U) \hat{\otimes} \mathcal{S}'(\mathbb{R}^{d+1})$ by setting

$$K_{j,s}(x, y) = \int_0^1 t^s K_{j-(d+2)}(x, y, t) \frac{dt}{t}, \qquad \Re s > 0. \tag{3.23}$$

As $K_{j-(d+2)}$ is smooth on $U \times (\mathbb{R}^{d+1} \setminus 0) \times \mathbb{R}$ the family $K_{j,s}$ is given on $U \times (\mathbb{R}^{d+1} \setminus 0)$ by a holomorphic family of smooth functions. Moreover the homogeneity of $K_{j-(d+2)}$ implies

(3.24)

$$\lambda^{d+2-(2z+j)} K_{j,s}(x, \lambda.y) - K_{j,s}(x, y) = \int_1^{\lambda^{-2}} t^s K_{j-(d+2)}(x, y, t) \frac{dt}{t}, \qquad \lambda > 0.$$

As the right-hand side is a holomorphic family of smooth functions we conclude that $K_{j,s}(x, y)$ is holomorphic family of almost homogeneous kernels with order $m_s + j$.

Finally the asymptotic expansion (3.22) implies that we have an asymptotic expansion $K_s \sim \sum K_{j,s}$ in the sense of holomorphic $\mathcal{K}^*$-valued families. So K_s is a holomorphic family of kernels and the proof is complete. ■

3.3 Complex powers of elliptic sublaplacians (general case)

Let $(M^{d+1}, \mathcal{V})$ be a compact Heisenberg manifold and let Δ be an elliptic sublaplacian on M. In this section we use the $\Psi_\mathcal{V}DO$ calculus with parameter of chapter 2 in order to construct as in [Se] the complex powers of Δ in such way to obtain a holomorphic 1-parameter group of $\Psi_\mathcal{V}DO$ operators.

By theorem 2.33 there exists $R > 0$ and an open pseudo-cone Λ containing $D(0, R) \setminus 0$, contained in $\Lambda_R = \{\lambda \in \mathbb{C} \setminus 0\ ;\ \Re\lambda < 0 \text{ or } |\lambda| < R\}$ and such that:

(i) for any $\lambda \in \Lambda$ the operator $\Delta - \lambda$ is invertible on $L^2(M)$;

(ii) the family $(\Delta - \lambda)^{-1}$, $\lambda \in \Lambda$, lies in $\Psi_\mathcal{V}^{-1,-2}(M, \Lambda)$;

(iii) for any pseudocone $\Lambda' \subset\subset \Lambda$ there exists $C_{\Lambda'} > 0$ such that

$$\|(\Delta - \lambda)^{-1}\| \leq C_{\Lambda'}(1 + |\lambda|)^{-1}, \qquad \lambda \in \Lambda'. \tag{3.25}$$

In particular each ray contained in Λ is a ray of minimal growth for Δ.

Suppose for simplicity that such a ray is the negative real axis. Then there exists a curve $\Gamma \subset \Lambda$ beginning at ∞, passing along the ray $\lambda < 0$ to a small circle about the origin with radius $\rho < R$, then clockwise around the circle and back to ∞ along the ray. For $\Re s < 0$ we set

$$\Delta_s = \frac{1}{2i\pi} \int_\Gamma \lambda^s (\Delta - \lambda)^{-1} d\lambda. \tag{3.26}$$

To define λ^s we choose a continuous determination of the logarithm on $\mathbb{C} \setminus \mathbb{R}_-$. This gives a determination of the argument which has shifted of 2π once λ has turned around the circle.

The above integral is convergent for the L^2-norm and yields a holomorphic family of bounded operators on $L^2(M)$. Indeed we have:

Lemma 3.18 *The family $(\Delta_s)_{\Re s<0}$ is a holomorphic family of $\Psi_{\mathcal{V}}DO$'s and* $\operatorname{ord}\Delta_s = 2s$.

Proof. Let φ, ψ be in $C^\infty(M)$ and with disjoint supports. As the family $(\Delta - \lambda)^{-1}$ lies in $\Psi_{\mathcal{V}}^{-1,-2}(M, \Lambda)$ proposition 2.11 implies that $\varphi(\Delta - \lambda)^{-1}\psi$ is a $\operatorname{Hol}^{-1}(\Lambda)$-family of smoothing operators. Thus $\varphi\Delta_s\psi$ is a holomorphic family of smoothing operators.

On the other hand we locally have

$$(\Delta - \lambda)^{-1} = f_{(\lambda)}(x, \sigma(x, D)) + R_{(\lambda)}, \tag{3.27}$$

where $f_{(\lambda)}$ is in $S^{-1,-2}(U \times \mathbb{R}^{d+1}, \Lambda)$ and $R_{(\lambda)}$ is a $\operatorname{Hol}^{-1}(\Lambda)$-family of smoothing operators on the open $U \subset \mathbb{R}^{d+1}$ equipped with a $\mathcal{V}$-frame. So on U we have

$$P_s = f_s(x, \sigma(x, D)) + R_s, \qquad \Re s < 0, \tag{3.28}$$

where R_s is a holomorphic family of smoothing operators and (f_s) is the holomorphic family of smooth functions on $U \times \mathbb{R}^{d+1}$ given by

$$f_s(x, \xi) = \int_\Gamma \lambda^s f_{(\lambda)}(x, \xi) d\lambda, \qquad \Re s < 0. \tag{3.29}$$

Thus it is enough to show that (f_s) is a holomorphic family of symbols.

Now we have an asymptotic expansion

$$f_{(\lambda)} \sim \sum_{j \geq 0} f_{(\lambda),-2-j}, \qquad f_{(\lambda),-2-j} \in S^{-1}_{-2-j}(U \times \mathbb{R}^{d+1}, \Lambda), \tag{3.30}$$

in the sense of symbols with parameters. For $j = 0, 1, \ldots$ we define a holomorphic C^∞-valued family by setting

$$f_{s,j} = \int_\Gamma \lambda^s f_{(\lambda),-2-j} d\lambda, \qquad \Re s < 0. \tag{3.31}$$

Let $t \in (0,1)$. As we don't change the value of the integral (3.31) by replacing Γ by $t^2\Gamma$ we have
(3.32)
$$f_{s,j}(x,t.\xi) = \int_{t^2\Gamma} \lambda^s f_{(\lambda),-2-j}(x,t.\xi)d\lambda = t^{-(s+2)} \int_{\Gamma} \lambda^s f_{(t^2\lambda),-2-j}(x,t.\xi)d\lambda.$$
So by almost homogeneity of $f_{(\lambda),-2-j}$ the family

$$f_{s,j}(x,t.\xi) - t^{2s-j} f_{s,j}(x,\xi), \qquad \Re s < 0, \tag{3.33}$$

is a holomorphic family with values in $S^{-\infty}(U \times \mathbb{R}^{d+1})$. Hence $(f_{s,j})$ is a holomorphic S^*_{ah}-valued family.

Finally the asymptotic expansion (3.30) implies that $f_s \sim \sum f_{j,s}$ in the sense of holomorphic family of symbols. Thus (f_s) is a holomorphic family of symbols and the proof is complete. ■

Lemma 3.19 ([Se]) *Suppose Δ invertible. Then the family $(\Delta_s)_{\Re s<0}$ has the following properties:*

1) It contains the negative integer powers of Δ, that is

$$\Delta_{-k} = \Delta^{-k} \qquad k \text{ integer} > 0. \tag{3.34}$$

2) It is a semi-group, i.e.

$$\Delta_s \Delta_t = \Delta_{s+t} \qquad \Re s < 0, \quad \Re t < 0. \tag{3.35}$$

Proof. We have

$$\Delta_{-k} = \frac{1}{2i\pi} \int_{\Gamma'} \lambda^{-k} (\Delta - \lambda)^{-1} d\lambda, \tag{3.36}$$

where Γ' is the circle of radius ρ traversed in the reverse, for the two integral along $\lambda \leq -\rho$ cancel each other. Then setting $\mu = \lambda^{-1}$ yields

$$\begin{aligned} \Delta_{-k} &= \frac{1}{2i\pi} \int_{\Gamma'^{-1}} \Delta^{-1} \mu^{k-1} (\mu - \Delta^{-1})^{-1} d\mu, \\ &= \Delta^{-1} (\Delta^{-1})^{k-1} = \Delta^{-k}. \end{aligned} \tag{3.37}$$

This follows from the Cauchy formula since the spectrum of Δ^{-1} lies inside Γ'^{-1}.

Now let Γ'' be a curve contained in Λ and enlacing Γ. As we don't change the value of the integral (3.26) by replacing Γ by Γ'' we get

$$\begin{aligned} \Delta_s \Delta_t &= \frac{-1}{4\pi^2} \int_{\Gamma''} \int_{\Gamma} \mu^s \lambda^t (\Delta - \mu)^{-1} (\Delta - \lambda)^{-1} d\mu d\lambda, \\ &\frac{-1}{4\pi^2} \int_{\Gamma''} \int_{\Gamma} \frac{\mu^s \lambda^t}{\mu - \lambda} ((\Delta - \mu)^{-1} - (\Delta - \lambda)^{-1}) d\mu d\lambda, \qquad (3.38) \\ &= \frac{-1}{4\pi^2} \int_{\Gamma''} \mu^s (\Delta - \mu)^{-1} \int_{\Gamma} \frac{\lambda^t d\lambda}{\lambda - \mu} d\mu - \frac{1}{4\pi^2} \int_{\Gamma} \lambda^t (\Delta - \lambda)^{-1} \int_{\Gamma''} \frac{\mu^s d\mu}{\mu - \lambda} d\lambda, \end{aligned}$$

By Cauchy formula the second integral in the last side vanishes and the first one is equal to

$$\frac{1}{2i\pi}\int_{\Gamma''}\mu^{s+t}(\Delta-\mu)^{-1}d\lambda=\Delta_{s+t}. \tag{3.39}$$

So $\Delta_s\Delta_t=\Delta_{s+t}$ and the proof is achieved. ■

We can now define the complex powers of Δ in the invertible case.

Definition 3.20 *Suppose Δ invertible. Then the complex power Δ^s for $s\in\mathbb{C}$ is defined by*

$$\Delta^s=\Delta^k\Delta_{s-k}, \tag{3.40}$$

where k is any integer $>\Re s$ whose value is irrelevant.

Combining lemma 3.18 and lemma 3.19 we obtain

Theorem 3.21 *Suppose Δ invertible. Then the family (Δ^s) of the complex powers of Δ given by definition 3.20 is a holomorphic 1-parameter group of $\Psi_{\mathcal{V}}DO$ operators containing $\Delta^0=1$ and $\Delta^1=\Delta$.*

Suppose now that Δ is selfadjoint and not invertible. Then the semigroup property (3.35) of Δ_s continue to hold and the equality (3.34) remains true if we replace the inverse by the partial inverse of Δ, i.e. the operator annihilating $\ker\Delta$ which is the inverse of Δ on $\operatorname{im}\Delta=(\ker\Delta)^{\perp}$. As $(\Delta^{-1})^k$ is the partial inverse of Δ^k the definition 3.20 still makes sense and we obtain:

Theorem 3.22 *Suppose Δ selfadjoint. Then the family (Δ^s) of the complex powers of Δ is a holomorphic 1-parameter group of $\Psi_{\mathcal{V}}DO$'s such that $\Delta^1=\Delta$ and $\Delta^0=1-\Pi_0$, where Π_0 is the orthogonal projection onto $\ker\Delta$.*

Remark 3.23 Suppose that Δ is selfadjoint and let $\lambda_k(\Delta)$ be the k'th eigenvalue of Δ counted with multiplicity. By corollary 2.32 and proposition 1.53 for k large we have $\lambda_k(\Delta)\sim\alpha k^{-\frac{d+2}{2}}$. So it is possible to characterize the Sobolev spaces of M in terms of Fourier series associate to an orthonormal basis of eigenvectors for Δ as in [Sh]. This enables us to relate the Sobolev regularity of $f(\Delta)$ to the polynomial growth of f. In particular if f has slow growth then $f(\Delta)$ maps continuously $C^\infty(M)$ to itself and if f has rapid decay then $f(\Delta)$ is a smoothing operator.

Remark 3.24 In the case of a pseudohermitian manifold (M,θ) the theorem 3.22 holds for the sublaplacian Δ_b and the conformal sublaplacian $\boxdot_\theta$. As the construction of the complex powers can as well be carried out for a sublaplacian acting on sections of a bundle, the theorem holds also for the Kohn Laplacian $\square_b$ acting on forms with elliptic bidegree.

Chapter 4

Non-commutative residue for Heisenberg manifolds

In this chapter we extend the trace on $\Psi_{\mathcal{V}}DO$'s with non-integral complex order as in [KV] and [CM2]. This new functionnal is holomorphic in the sense that the evaluation on any holomorphic family of $\Psi_{\mathcal{V}}DO$'s of non-integral order defines a holomorphic function, and we show that it gives rise to a residue trace on $\Psi_{\mathcal{V}}DO$ with integral order which is an analogue of the non-commutative residue for $\Psi_{\mathcal{V}}DO$ operators (section 4.1, theorem 4.5 and proposition 4.9). Then we prove that this new non-commutative residue extends the Dixmier trace on the $\Psi_{\mathcal{V}}DO$ algebra (section 4.2, theorem 4.11) and is the unique trace up to a constant multiple on this algebra quotiented by the smoothing operators(section 4.3, theorem 4.15). As corollary we obtain a complete characterization of sums of commutators in the $\Psi_{\mathcal{V}}DO$ algebra (corollary 4.16).

4.1 Trace regularization and non-commutative residue

Let $(M^{d+1}, \mathcal{V})$ be a compact Heisenberg manifold and $\mathcal{E}$ a vector bundle over M. If P is a $\Psi_{\mathcal{V}}DO$ of integrable order, i.e. $\Re \mathrm{ord} P < -(d+2)$, its kernel is continuous so that P is traceable and its trace is given by

$$\mathrm{Trace}(P) = \int_M \mathrm{tr}_{\mathcal{E}}\, k_P(x,x)dx. \tag{4.1}$$

Following closely [KV] and [CM2] we shall show that the functional Trace *a priori* defined on

$$\Psi_{\mathcal{V}}^{\mathrm{int}}(M,\mathcal{E}) = \{P \in \Psi_{\mathcal{V}}^{*}(M,\mathcal{E}); \Re \mathrm{ord} P < -(d+2)\}, \tag{4.2}$$

can be holomorphically extended to a trace functional TR to

$$\Psi_{\mathcal{V}}^{\mathbb{C}\setminus\mathbb{Z}} = \{P \in \Psi_{\mathcal{V}}^{*}(M,\mathcal{E}); \mathrm{ord} P \in \mathbb{C}\setminus\mathbb{Z}\}. \tag{4.3}$$

Moreover this trace gives rise to a residue trace on $\Psi^{\mathbb{Z}}_{\mathcal{V}}(M)$ which turns out to be the complete analogue of the non-commutative residue for Heisenberg manifolds.

The starting point is to reinterpret the lemma 1.23 in terms of holomorphic maps.

Lemma 4.1 *For $f \in S_{\mathbb{C}\mathbb{Z}}(\mathbb{R}^{d+1})$ denote by τ_f its unique homogeneous extension as a tempered distribution on $\mathbb{R}^{d+1}$ given by lemma 1.23. Then:*

1) The map $f \to \tau_f$ is holomorphic from $S_{\mathbb{C}\mathbb{Z}}(\mathbb{R}^{d+1})$ into $\mathcal{S}'(\mathbb{R}^{d+1})$, i.e. for any holomorphic $S_{\mathbb{C}\mathbb{Z}}(\mathbb{R}^{d+1})$-valued family (f_z) the family (τ_{f_z}) is a holomorphic family of tempered distributions.

2) Let (f_z) be a holomorphic family of homogeneous symbols defined near the integer m and such that $\mathrm{ord} f_z = z$. Then for any $u \in \mathcal{S}(\mathbb{R}^{d+1})$ the function $z \to \langle \tau_{f_z}, u\rangle$ has only meromorphic singularities near m with at most simple poles with residues

$$\mathrm{res}_{z=k}\langle \tau_{f_z}, u\rangle = \sum_{<\alpha>=-(m+d+2)} \frac{1}{\alpha!} c_\alpha(f_m) u^{(\alpha)}(0), \tag{4.4}$$

where, as in lemma 1.23, the constants $c_\alpha(f_k)$ are given by

$$c_\alpha(f_k) = \frac{1}{\alpha!}\int_{\|\xi\|=1} \xi^\alpha f_k(\xi) i_E d\xi. \tag{4.5}$$

Proof. Let f_z be a holomorphic family of symbols on $\mathbb{R}^{d+1}$ with non integral order m_z and let τ_z be the unique homogeneous extension of f_z as a tempered distribution on $\mathbb{R}^{d+1}$.

For $\Re m_z > -(d+2)$ the symbol f_z is integrable near the origin and defines a distribution which is its unique homogeneous extension. Then (τ_z) is a holomorphic family of tempered distributions. Thus, shrinking Ω if necessary, we can suppose that m_z stays in some stripe $\{|\Re z - m| < 1\}$ with m integer $\leq -(d+2)$. Then in the definition (1.79) of τ_z we maay take $k = -(m+d+2)$ and get

(4.6)
$$\langle \tau_z, u\rangle = \int (u(\xi) - \sum_{<\alpha>\leq -(m+d+2)} \frac{\xi^\alpha}{\alpha!} u^{(\alpha)}(0) \psi_z(\|\xi\|)) f_z(\xi) d\xi, \qquad u\mathcal{S}(\mathbb{R}^{d+1}),$$

with $\psi_z \in C^\infty_c([0,\infty))$ equals to 1 near zero and satisfying

$$\int \mu^a \psi'_z(\mu) \frac{d\mu}{\mu} = 0 \qquad \text{for } a = m_z - (m+d+2), \ldots, m_z - m. \tag{4.7}$$

For instance we can take $\psi_z(\mu)$ of the form

$$\psi_z(\mu) = \int_{\log \mu}^{+\infty} h_z(s) ds, \qquad h_z(t) = \prod_{j=m}^{-(d+2)} (\frac{1}{m_z - j}\frac{d}{dt} + 1) g(t), \tag{4.8}$$

where g is some compactly supported function on $\mathbb{R}$ such that $\int g(t)dt = 1$. Remember that since m_z is not an integer the distribution τ_z is uniquely defined and independent of the above choices. Then the family $(\psi_z(\|.\|))$ is a holomorphic family of smooth functions supported in a fixed compact set and identically equal to 1 in a fixed neighborhood of the origin. Thus the formula (4.6) shows that $(\tau_z$ is a holomorphic family of tempered distributions.

Suppose now that the family (f_z) is holomorphic around m with order $m_z = z$ and let us investigate the singularity of $\langle \tau_z, u\rangle$ near $z = m$. This singularity comes only from the appearance of the term $\frac{1}{z-m}$ in the equality (4.8). We can isolate it if we write h_z in the form

$$(4.9)\quad h_z(t) = \frac{1}{z-m}\frac{d}{dt}k_z(t) + k_z(t), \qquad k_z(t) = \prod_{j=m+1}^{-(d+2)}(\frac{1}{z-j}\frac{d}{dt} + 1)g(t).$$

Then

$$(4.10)\qquad \psi_z(\mu) = \frac{-1}{z-m}k_z(\log\mu) + \varphi_z(\mu), \qquad \varphi_z(\mu) = \int_{\log\mu}^{+\infty} k_z(t)dt,$$

and (4.6) becomes

$$\begin{aligned}\langle \tau_z, u\rangle &= \int (u(\xi) - \sum_{<\alpha>\leq -(m+d+2)} \frac{\xi^\alpha}{\alpha!}u^{(\alpha)}(0)\varphi_z(\|\xi\|))f_z(\xi)d\xi \\ (4.11)\qquad &- \frac{1}{z-m}\sum_{<\alpha>\leq -(m+d+2)} \frac{u^{(\alpha)}(0)}{\alpha!}\int \xi^\alpha k_z(\log\|\xi\|)f_z(\xi)d\xi.\end{aligned}$$

As $(\varphi_z(\|.\|))$ is near $z = m$ a holomorphic family of smooth functions supported in a fixed compact set and identically equal to 1 in a fixed neighborhood of the origin, the first integral in the above right-hand side defines a holomorphic function near $z = m$.

Similarly, the family (k_z) is a holomorphic family of smooth functions on $\mathbb{R}$ supported on a fixed compact set and the integrals $\int \xi^\alpha k_z(\log\|\xi\|)f_z(\xi)d\xi$ defines holomorphic functions. It follows then that $\langle \tau_z, u\rangle$ has at most a simple singularity near $z = m$ with residue

$$\begin{aligned}&\sum_{<\alpha>\leq -(m+d+2)} \frac{u^{(\alpha)}(0)}{\alpha!}\int \xi^\alpha k_m(\log\|\xi\|)f_m(\xi)d\xi \\ (4.12)\quad = &\sum_{<\alpha>\leq -(m+d+2)} \frac{u^{(\alpha)}(0)}{\alpha!}\int_0^\infty \mu^{<\alpha>+m+d+2}k_m(\log\mu)\frac{d\mu}{\mu}c_\alpha(f_m).\end{aligned}$$

However it follows from (4.9) that k_m satisfies

$$(4.13)\qquad \int \mu^a k_m(\log\mu)\frac{d\mu}{\mu} = 0, \qquad a = 1,\dots,-(m+d+2),$$

$$(4.14) \qquad \int_0^\infty k_m(\log\mu)\frac{d\mu}{\mu} = \int_{-\infty}^{+\infty} k_m(t)dt = \int_{-\infty}^{+\infty} g(t)dt = 1.$$

Therefore $\mathrm{res}_{z=m}\langle\tau_z, u\rangle = -\sum_{<\alpha>=-(m+d+2)} \frac{u^{(\alpha)}(0)}{\alpha!} c_\alpha(f_m)$ and the proof is complete. ■

Following [CM2] we consider the functional

$$(4.15) \qquad L(f) = \int f(\xi)d\xi, \qquad f \in S^{\mathrm{int}}(\mathbb{R}^{d+1}).$$

defined on

$$(4.16) \qquad S^{\mathrm{int}}(\mathbb{R}^{d+1}) = \{f \in S^*(\mathbb{R}^{d+1}); \Re\mathrm{ord} f < -(d+2)\}.$$

Lemma 4.2 *Let L be the above functional on $S^{int}(\mathbb{R}^{d+1})$. Then:*

1) L has an unique holomorphic extension $\tilde{L}$ on $S^{\mathbb{C}\setminus\mathbb{Z}}(\mathbb{R}^{d+1})$, in the sense that for any holomorphic family of symbols (f_z) with values in $S^{\mathbb{C}\setminus\mathbb{Z}}(\mathbb{R}^{d+1})$ the function $\tilde{L}(f_z)$ is holomorphic. The value of $\tilde{L}$ on a symbol $f \sim \sum f_{m-j}$ with non integral order is given by

$$(4.17) \qquad \tilde{L}(f) = \int (f(\xi) - \sum_{j\leq N} \tau_{m-j}(\xi))d\xi, \qquad N \geq \Re m + d + 2,$$

where τ_{m-j} is the unique homogeneous extension of f_{m-j} provided by lemma 1.23.

2) Let (f_z) be a holomorphic $S^*(\mathbb{R}^{d+1})$-valued family such that $\mathrm{ord} f_z = z$. Then $\tilde{L}(f_z)$ has at most simple pole singularities near $\mathbb{Z}$ with residues

(4.18)

$$\mathrm{res}_{z=k}\, \tilde{L}(f_z) = -c_0(f_{k,-(d+2)}) = -\int_{\|\xi\|=1} f_{k,-(d+2)}(\xi) i_E d\xi, \qquad k \in \mathbb{Z}.$$

Proof. First the extension is necessarily unique since the functional L is holomorphic on $S^{\mathrm{int}}(\mathbb{R}^{d+1})$ and each $f \in S^{\mathbb{C}\setminus\mathbb{Z}}(\mathbb{R}^{d+1})$ can be connected to $S^{\mathrm{int}}(\mathbb{R}^{d+1})$ by a holomorphic path within $S^{\mathbb{C}\setminus\mathbb{Z}}(\mathbb{R}^{d+1})$.

Let $f \sim \sum f_{m-j}$ be a symbol with non-integral order and denote by τ_{m-j} the unique homogeneous extension of f_{m-j} given by lemma 1.23. For $N \geq \Re m + d + 2$ the distribution $f - \sum_{j\leq N} \tau_{m-j}$ agrees with an integrable function near ∞ and we can set

$$(4.19) \qquad \tilde{L}(f) = (f - \sum_{j\leq N} \tau_{m-j})^\wedge(0) = \int (f - \sum_{j\leq N} \tau_{m-j})(\xi)d\xi.$$

In fact if $j > \Re m + d + 2$ then τ_{m-j} is also integrable near ∞ and we can define $\hat{\tau}_{m-j}(0)$. By homogeneity the value must be 0, so the value of the integer N is irrelevant and (4.19) defines a functional on $S^{\mathbb{C}\mathbb{Z}}(\mathbb{R}^{d+1})$ which agrees with L on symbols of integrable non-integral order.

Let us show that $\tilde{L}$ is holomorphic and let $f_z(\xi) \sim \sum f_{z,m_z-j}(\xi)$ be a holomorphic family of symbols with non-integral order m_z. As $\tilde{L}$ agrees with L on $S^{\text{int}}(\mathbb{R}^{d+1}) \cap S^{\mathbb{C}\mathbb{Z}}(\mathbb{R}^{d+1})$ we can suppose that m_z lies in some stripe $\{|\Re z - k| < 1\}$ with k integer $\geq -(d+2)$. Then in (4.19) we can set $N = k+d+2$, and if we pick some function $\varphi \in C_c^\infty(\mathbb{R}^{d+1})$ such that $\varphi = 1$ near the origin we obtain

$$\begin{aligned} \tilde{L}(f_z) &= \int (f_z(\xi) - (1-\varphi(\xi) \sum_{j\leq N} f_{z,m_z-j}(\xi))d\xi - \sum_{j\leq N} (\varphi \tau_{z,m_z-j})^\wedge(0), \\ &= L(f_z - (1-\varphi) \sum_{j\leq N} f_{z,m_z-j}) - \sum_{j\leq N} \langle \tau_{z,m_z-j}, \varphi \rangle. \end{aligned} \tag{4.20}$$

The functions $\langle \tau_{z,m_z-j}, \varphi \rangle$ are holomorphic by lemma 4.1 and $f_z - (1-\varphi) \sum_{j\leq k+d+2} f_{z,m_z-j}$ is a holomorphic family of integrable symbols. Therefore $\tilde{L}(f_z)$ is a holomorphic map and we have shown that $\tilde{L}$ is a holomorphic functional.

Finally suppose that (f_z) is a holomorphic family of symbols near $z = k$ such that $\text{ord} f_z = z$ Then $L(f_z - (1-\varphi) \sum_{j\leq N} f_{z,j})$ is holomorphic near $z = k$ and by lemma 4.1 the function $\sum \langle \tau_{z,j}, \varphi \rangle$ has at most a simple pole singularity at $z = k$ with a residue equal to $c_0(f_{k,-(d+2)})$. This concludes the proof. ■

Remark 4.3 Let (f_z) be a holomorphic family of symbols around $z = 0$ such that $\text{ord} f_z = z$. If $c_0(f_{0,-(d+2)}) = 0$ the function $\tilde{L}(f_z)$ is regular at zero, but by (4.11) and (4.11) the regular value depends on both the values of $f_{0,-(d+2)}$ and $\partial_z f_{0,-(d+2)}$.

As the proof works as well with smooth families of symbols we obtain:

Lemma 4.4 *Let U be an open subset of $\mathbb{R}^{d+1}$.*

> *1) The map $f \to \tilde{L}(f(x,.))$ is holomorphic from $S^{\mathbb{C}\mathbb{Z}}(U \times \mathbb{R}^{d+1})$ into $C^\infty(U)$.*
>
> *2) Let (f_z) be a holomorphic $S^*(U \times \mathbb{R}^{d+1})$-valued family such that $\text{ord} f_z = z$. Then $\tilde{L}(f_z(x,.))$ is meromorphic for the C^∞-topology.*

Theorem 4.5 *Let $(M^{d+1}, \mathcal{V})$ be a compact Heisenberg manifold and let $\mathcal{E}$ be a vector bundle over M.*

> *1) The functional* Trace *on $\Psi_{\mathcal{V}}^{\text{int}}(M,\mathcal{E})$ has an unique holomorphic extension on $\Psi_{\mathcal{V}}^{\mathbb{C}\mathbb{Z}}(M,\mathcal{E})$ defined by*
>
> $$\text{TR}\, P = \int_M \text{tr}_{\mathcal{E}}\, t_P(x), \qquad P \in \Psi_{\mathcal{V}}^{\mathbb{C}\mathbb{Z}}(M,\mathcal{E}), \tag{4.21}$$

where $t_P(x)$ is a $\operatorname{END}\mathcal{E}$*-valued density on M invariant by Heisenberg diffeomorphisms.*

2) Let P_1 and P_2 be in $\Psi_{\mathcal{V}}^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})$ and such that $\operatorname{ord}P_1 + \operatorname{ord}P_2 \notin \mathbb{Z}$. *Then*

$$\operatorname{TR} P_1P_2 = \operatorname{TR} P_2P_1. \tag{4.22}$$

3) Let $P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E})$ and let (P_z) be a holomorphic family of $\Psi_{\mathcal{V}}DO$'s such that $P_0 = P$ and $\operatorname{ord}P_z = z + \operatorname{ord}P$. Then $\operatorname{TR} P_z$ has at most a simple pole at $z = 0$ and we have

$$\operatorname{res}_{z=0} \operatorname{TR} P_z = -\int_M \operatorname{tr}_{\mathcal{E}} c_P(x), \tag{4.23}$$

where $c_P(x)$ is the density on M which occurs as the coefficient of the logarithmic singularity of the kernel of P near the diagonal (cf. proposition 1.43).

Proof. First a holomorphic extension of Trace is necessarily unique as any $P \in \Psi_{\mathcal{V}}^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})$ can be connected to $\Psi_{\mathcal{V}}^{\text{int}}(M,\mathcal{E})$ by means of the holomorphic path $z \to \Delta^{z/2}P$, where Δ is an elliptic selfdajoint sublaplacian on M.

If P is a $\Psi_{\mathcal{V}}DO$ acting on $C^\infty(M,\mathcal{E})$ with integrable order, the restriction of its kernel on the diagonal $k_P(x,x)$ is a smooth density with values in $\operatorname{END}\mathcal{E}$ and we have

$$\operatorname{Trace} P = \int_M k_P(x,x). \tag{4.24}$$

The map $P \to k_P(x,x)$ is holomorphic from $\Psi_{\mathcal{V}}^{\text{int}}(M,\mathcal{E})$ into the space of $\operatorname{END}\mathcal{E}$-valued densities. The strategy of the proof is to construct an analytic extension of this map on $\Psi_{\mathcal{V}}^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})$. Then an integration over M would give the required holomorphic extension of Trace.

Actually, using a partition of unity it is enough to proceed locally and we can restrict ourselves to the case of scalar $\Psi_{\mathcal{V}}DO$'s on an open subset U of $\mathbb{R}^{d+1}$ with a $\mathcal{V}$-frame. Such an operator is of the form

$$P = f(x,\sigma(x,D)) + R, \tag{4.25}$$

with $f \in S^*(U \times \mathbb{R}^{d+1})$ and R smoothing. So lemma 4.4 provides us with an analytic continuation of $P \to k_P(x,x)$ on $\Psi_{\mathcal{V}}^{\mathbb{C}\setminus\mathbb{Z}}(U)$ by letting

$$t_P(x) = (2\pi)^{-(d+2)}|\varepsilon'_x|\tilde{L}(f(x,.)) + k_R(x,x). \tag{4.26}$$

Note that the definition is independent of the choice of f and R. Moreover, as $k_P(x,x)$ is a density and by proposition 3.14 the action of a Heisenberg diffeomorphism on $\Psi_{\mathcal{V}}DO$'s is holomorphic, we get a holomorphic extension at the level of $\operatorname{END}\mathcal{E}$-valued densities on M. Then the holomorphic extension of Trace on $\Psi_{\mathcal{V}}^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})$ is given by

$$\text{(4.27)} \qquad \operatorname{TR} P = \int_M \operatorname{tr}_{\mathcal{E}} t_P(x), \qquad P \in \Psi_{\mathcal{V}}^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E}).$$

Now, let $P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(U)$ and let (P_z) be a holomorphic family of $\Psi_{\mathcal{V}}DO$'s such that $P_0 = P$ and $\operatorname{ord} P_z = z + \operatorname{ord} P$. It follows also from lemma 4.4 that locally $t_{P_z}(x)$ has at most a simple pole at $z = 0$ with a residue equal to

$$\text{(4.28)} \qquad -(2\pi)^{-(d+2)}|\varepsilon'_x| c_0(f_{-(d+2)}(x,.)) = -c_P(x),$$

where f_{d+2} is the symbol of degree $-(d+2)$ of P. Thus $\operatorname{TR} P_z$ has at most a simple pole with a residue equal to $-\int_M \operatorname{tr}_{\mathcal{E}} c_P(x)$.

Finally, to see that TR is a trace let P_1 and P_2 be in $\Psi_{\mathcal{V}}^{*}(M,\mathcal{E})$ such that $m = \operatorname{ord} P_1 + \operatorname{ord} P_2$ is not an integer. If Δ is an elliptic selfadjoint sublaplacian we have

$$\text{(4.29)} \qquad \operatorname{Trace} P_1 P_2 \Delta^{-z/2} = \operatorname{Trace} P_2 \Delta^{-z/2} P_1, \qquad z + m \notin \mathbb{Z},$$

for this is true for $\Re z + m < -(d+2)$. Setting $z = 0$ we obtain $\operatorname{TR} P_1 P_2 = \operatorname{TR} P_2 P_1$. ■

Remark 4.6 As we worked at the level of densities the theorem rephrased only in terms of densities continues to hold for non-compact manifolds.

Remark 4.7 Let $P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E})$ be such that $c_P(x) = 0$. For any holomorphic family (P_z) of $\Psi_{\mathcal{V}}DO$'s near $z = 0$ such that $P_0 = P$ and $\operatorname{ord} P_z = z + \operatorname{ord} P$ the function $\operatorname{TR} P_z$ is regular at $z = 0$. By remark 4.3 the regular value depends on the choice of the family (P_z), but if $(P_{1,z})$ and $(P_{2,z})$ are two such families satisfying furthermore $\operatorname{ord} P_{1,z} - \operatorname{ord} P_{2,z} < \operatorname{ord} P$ then the regular values at zero of $\operatorname{TR} P_{1,z}$ and $\operatorname{TR} P_{2,z}$ coincides.

We define the non-commutative residue for Heisenberg manifolds as follows:

Definition 4.8 *Let $(M,\mathcal{V})$ be a compact Heisenberg manifold. The non-commutative residue on $\Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E})$ is the linear functional defined by*

$$\text{(4.30)} \qquad \operatorname{Res} P = \int_M c_P(x), \qquad P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E}),$$

where $c_P(x)$ is the density on M which occurs as the coefficient of the logarithmic singularity of the kernel of P near the diagonal.

Proposition 4.9 *Let $(M,\mathcal{V})$ be a compact Heisenberg manifold.*

1) Let $P \in \Psi^{\mathbb{Z}}_{\mathcal{V}}(M,\mathcal{E})$ and let (P_z) be a holomorphic family of $\Psi^{*}_{\mathcal{V}}(M,\mathcal{E})$ such that $P_0 = P$ and $\operatorname{ord} P_z = z + \operatorname{ord} P$. Then

$$\operatorname{Res} P = -\operatorname{res}_{z=0} \operatorname{TR} P_z. \tag{4.31}$$

In particular if Δ is an elliptic sublaplacian on M we have

$$\operatorname{Res} P = \operatorname{res}_{z=0} \operatorname{TR} P\Delta^{-z/2}, \qquad P \in \Psi^{\mathbb{Z}}_{\mathcal{V}}(M,\mathcal{E}). \tag{4.32}$$

2) The functional Res *is a trace on $\Psi^{\mathbb{Z}}_{\mathcal{V}}(M,\mathcal{E})$ vanishing on $\Psi_{\mathcal{V}}DO$ operators with integral order $< -(d+2)$.*

3) Let $\phi : (M,\mathcal{V}) \to (\tilde{M},\tilde{\mathcal{V}})$ be a Heisenberg diffeomorphism. Then

$$\operatorname{Res} \phi_* P = \operatorname{Res} P, \qquad P \in \Psi^{\mathbb{Z}}_{\mathcal{V}}(M,\mathcal{E}). \tag{4.33}$$

Proof. The first assertion is just a restatement of the last assertion in theorem 4.5, and by definition Res vanishes on $\Psi_{\mathcal{V}}DO$'s with integral order $< -(d+2)$.

Moreover if P_1, P_2 are in $\Psi^{\mathbb{Z}}_{\mathcal{V}}(M,\mathcal{E})$ we can pick an elliptic sublaplacian Δ on M and obtain

$$\operatorname{Res} P_1P_2 = -\operatorname{res}_{z=0} \operatorname{TR} P_1P_2\Delta^{-z/2} = -\operatorname{res}_{z=0} \operatorname{TR} P_2\Delta^{-z/2}P_1 = \operatorname{Res} P_2P_1. \tag{4.34}$$

So Res is a trace on $\Psi^{\mathbb{Z}}_{\mathcal{V}}(M,\mathcal{E})$.

Finally let $\phi : (M,\mathcal{V}) \to (\tilde{M},\tilde{\mathcal{V}})$ be a Heisenberg diffeomorphism and let $P \in \Psi^{\mathbb{Z}}_{\mathcal{V}}(M,\mathcal{E})$. By proposition 1.43 ϕ_*P lies in $\Phi_{\tilde{\mathcal{V}}}(\tilde{M},\phi_*\mathcal{E})$ and we have $c_{\phi_*P}(\tilde{x}) = \phi_*(c_P(x))$. Hence $\operatorname{Res} \phi_*P = \operatorname{Res} P$. ■

4.2 The Dixmier trace of $\Psi_{\mathcal{V}}DO$ operators

Let $(M^{d+2},\mathcal{V})$ be a compact Heisenberg manifold. In this section we shall prove that the non-commutative residue agrees with the Dixmier trace on $\Psi_{\mathcal{V}}DO$ operators of order $\leq -(d+2)$. Then we will get an analogue of the following theorem for classical pseudodifferential operators.

Theorem 4.10 ([Co3]) *Let M^d be a compact manifold and $\mathcal{E}$ a vector bundle over M.*

1) For any $P \in \Psi^{*}(M,\mathcal{E})$ with integral order $-k < 0$ we have

$$\mu_n(P) = \mathrm{O}(n^{-\frac{k}{d}}) \qquad \text{as } n \to \infty, \tag{4.35}$$

where $\mu_n(P)$ is the $(n+1)$'th characteristic value of P.

2) If P has integral order $\leq -d$, it is measurable for the Dixmier trace and we have

$$\int\!\!\!\!\!\!- P = \frac{1}{d} \operatorname{Res} P, \tag{4.36}$$

where Res *denotes the non-commutative residue on* $\Psi^{\mathbb{Z}}(M, \mathcal{E})$.

Before enouncing the corresponding theorem for Heisenberg manifolds, let us briefly recall the definition and the main properties of the Dixmier trace (for more details see [Co1] and [CM2]).

Suppose $\mathcal{H}$ is a separable Hilbert space and let $\mathcal{K}$ be its ideal of compact operators. If T is a compact operator the characteristic value $\mu_n(T)$ is the $(n+1)$'th eigenvalue of $|T| = (T^*T)^{1/2}$. Then one can show that

$$\begin{aligned} \mu_n(T) &= \inf\{\|T_{E^\perp}\|; \dim E = n\}, \\ &= \operatorname{dist}(T, \mathcal{R}_n), \qquad \mathcal{R}_n = \{\text{operators of rank} \leq n\}, \end{aligned} \tag{4.37}$$

the first equality being the max-min principle. This implies

$$\mu_n(ATB) \leq \|A\| \mu_n(T) \|B\| \qquad \text{for } A, B \in \mathcal{L}(\mathcal{H}). \tag{4.38}$$

The compact operator T lies in $\mathcal{L}^1$, i.e. T is traceable, if and only if we have

$$\|T\|_1 = \sum_{n=0}^{\infty} \mu_n(T) < \infty. \tag{4.39}$$

Then the trace of T is given by

$$\operatorname{Trace} T = \sum_{n=0}^{\infty} \langle T\xi_n | \xi_n \rangle, \qquad (\xi_n) \text{ orthonormal basis}, \tag{4.40}$$

the value of the sum being independent of the choice of the orthonormal basis.

The Dixmier trace arises in the study of the divergency of the trace of a positive operator only satisfying

$$\mu_n(T) = \mathrm{O}(1/n) \qquad \text{as } n \to \infty. \tag{4.41}$$

Define the partial sums

$$\sigma_N(T) = \sum_{n=0}^{N-1} \mu_n(T), \qquad T \in \mathcal{K}. \tag{4.42}$$

If $\mu_n(T) = \mathrm{O}(1/n)$ then $\sigma_N(T) = \mathrm{O}(\log N)$. The domain of the Dixmier trace is the two-sided ideal

$$\mathcal{L}^{(1,\infty)} = \{T \in \mathcal{K}; \|T\|_{(1,\infty)} < \infty\}, \qquad \|T\|_{(1,\infty)} = \sup \frac{\sigma_N(T)}{\log N}. \tag{4.43}$$

We can define $\sigma_N(T)$ for non integer values of N by means of the interpolation formula

(4.44) $$\sigma_\lambda(T) = \inf\{\|x\|_1 + \lambda\|y\|; x + y = T\}, \qquad \lambda > 0,$$

which in turn shows that $\mathcal{L}^{(1,\infty)}$ is the (real) interpolated space of $\mathcal{L}^1$ and $\mathcal{K}$. We define the Cesāro mean of $\sigma_\lambda(T)$ by setting

(4.45) $$\tau_\lambda(T) = \frac{1}{\log\lambda}\int_e^\lambda \frac{\sigma_u(T)}{\log u}\frac{du}{u}, \qquad \lambda \geq e.$$

The functionals τ_λ have the asymptotic additivity property

(4.46) $$|\tau_\lambda(T_1+T_2)-\tau_\lambda(T_1)-\tau_\lambda(T_2)| \leq 3(\|T_1\|_{(1,\infty)}+\|T_2\|_{(1,\infty)})\frac{\log\log\lambda}{\log\lambda}, \qquad T_j \geq 0.$$

It follows that any limit point $\lim_\omega \tau_\lambda$ of this functional gives rise to a positive continuous trace on $\mathcal{L}^{(1,\infty)}$, denoted Tr_ω, such that

(4.47) $$\mathrm{Tr}_\omega T = \lim_\omega \tau_\lambda(T), \qquad T \in \mathcal{L}^{(1,\infty)},\ T \geq 0.$$

Moreover, if S is a (topological) isomorphism from $\mathcal{H}$ onto another Hilbert space $\mathcal{H}'$, then we have

(4.48) $$\mathrm{Tr}_{\omega\mathcal{H}'}(STS^{-1}) = \mathrm{Tr}_{\omega\mathcal{H}} T, \qquad T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$

So Tr_ω does not depend on the choice of an inner product on $\mathcal{H}$.

In fact the choice of the limit procedure $\lim_\omega$ is not important because in most examples the value $\mathrm{Tr}_\omega T$ is independent of the choice of the limit procedure. An operator for which this property occurs is said to be *measurable* and then we let

(4.49) $$⨍ T = \mathrm{Tr}_\omega T.$$

If $T \in \mathcal{L}^{(1,\infty)}$ is positive, T is measurable if, and only if,

(4.50) $$\lim_{\lambda\to\infty} \tau_\lambda(T) \text{ exists},$$

and then the Dixmier trace of T is equal to the value of this limit. So if T is a positive compact operator such that

(4.51) $$\frac{1}{\log N}\sum_{n=0}^{N-1}\mu_n(T) \to L \qquad \text{as } N \to \infty$$

then T is measurable and $⨍ T = L$. In particular the Dixmier trace vanishes on finite rank operators. Hence it vanishes on their closure in $\mathcal{L}^{(1,\infty)}$, the ideal

(4.52) $$\mathcal{L}_0^{(1,\infty)} = \{T \in \mathcal{K}; \sigma_N(T) = \mathrm{o}(\log N)\},$$

which contains the ideal $\mathcal{L}^1$ of trace-class operators.

An important example of measurable operator for which the Dixmier trace can be computed is provided by an application the Tauberian theorem of Hardy-Littlewood [Ha]. If T is a compact operator such that $\mu_n(T) = \mathrm{O}(\frac{1}{n})$ and

$$(s-1)\sum_{n\geq 0}\mu_n(T)^s \to L \qquad \text{as } s\to 1^+, \tag{4.53}$$

then T is measurable and $⨍ T = L$.

For instance, if Δ is an elliptic selfadjoint sublaplacian on a compact Heisenberg manifold $(M^{d+1},\mathcal{V})$ then using theorem 4.5 we see that $\Delta^{-\frac{d+2}{2}}$ is measurable and

$$⨍ \Delta^{-\frac{d+2}{2}} = \frac{1}{d+2}\operatorname{res}\Delta^{-\frac{d+2}{2}}. \tag{4.54}$$

Indeed the following holds:

Theorem 4.11 *Let $(M^{d+1},\mathcal{V})$ be a compact Heisenberg manifold and let $\mathcal{E}$ be a vector bundle over M.*

1) Let $P\in\Psi_{\mathcal{V}}^m(M,\mathcal{E})$ with $-k=\Re m<0$. Then we have

$$\mu_n(P) = \mathrm{O}(n^{-\frac{k}{d+2}}) \qquad \text{as } n\to\infty. \tag{4.55}$$

2) Each $P\in\Psi_{\mathcal{V}}^{-(d+2)}(M,\mathcal{E})$ is measurable for the Dixmier trace and we have

$$⨍ P = \frac{1}{d+2}\operatorname{Res} P, \tag{4.56}$$

where Res *is the non-commutative residue on $\Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E})$.*

Proof. First let Δ be a selfadjoint elliptic sublaplacian on M and let $\lambda_n(\Delta)$ be the $(n+1)$'th eigenvalue of Δ counted with multiplicity. By proposition 1.53 for n large we have

$$\lambda_n(\Delta)\sim(An)^{\frac{d+2}{2}}, \qquad A>0. \tag{4.57}$$

As $P\Delta^{-k/2}$ is bounded the inequality (4.38) implies

$$\mu_n(P) = \mu_n(P\Delta^{-k/2}\Delta^{k/2}) \leq \|\Delta^{-k/2}\|\lambda_n(\Delta)^{-k/2} = \mathrm{O}(n^{-\frac{k}{d+2}}). \tag{4.58}$$

In particular if P has order $\leq -(d+2)$ then $\mu_n(P)=\mathrm{O}(n^{-1})$ and P belongs to $\mathcal{L}^{(1,\infty)}(L^2(M))$, i.e. it is in the domain of the Dixmier trace.

Let us now show that for any limit procedure $\lim_\omega$ as before we have

$$\operatorname{Tr}_\omega P = \frac{1}{d+2} \operatorname{Res} P. \tag{4.59}$$

In fact both sides vanish on smoothing operators and it follows from the equality (4.48) and proposition 1.43 that they are both invariant by Heisenberg diffeomorphisms. So it is enough to check (4.59) locally, and we can restrict ourselves to the case of scalar ΨDO's compactly supported in a trivializing Heisenberg chart which is diffeomorphic to $\mathbb{R}^{d+1}$. Then $\mathbb{R}^{d+1}$ inherits a $\mathcal{V}$-frame and we can identify compactly supported $\Psi_{\mathcal{V}}DO$'s on $\mathbb{R}^{d+1}$ with $\Psi_{\mathcal{V}}DO$'s on M. This will allow to perform a yoga between $\Psi_{\mathcal{V}}DO$'s M and those in $\mathbb{R}^{d+1}$.

However each $P \in \Psi_{\mathcal{V},\text{comp}}(\mathbb{R}^{d+1})$ can be written P as

$$P = P_{c_P} + P' + P", \tag{4.60}$$

where P_{c_P}, P', and P'' are compactly supported $\Psi_{\mathcal{V}}DO$'s such that

- P_{c_P} has kernel $-c_P \log \|\varepsilon_x(y)\|'$,
- P' has a kernel $|\varepsilon'_x| a(x, -\varepsilon_x(y))$ with $a \in \mathcal{K}_0(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ homogeneous of degree 0 in the last variable,
- P'' has order $\leq -(d+3)$.

Obviously $\operatorname{Tr}_\omega P'' = \operatorname{Res} P'' = \operatorname{Res} P' = 0$. Moreover the holomorphic family (P_z) of $\Psi_{\mathcal{V}}DO$'s given around $z=0$ by the kernels

$$k_z(x,y) = |\varepsilon'_x| a(x, -\varepsilon_x(y)) \|\varepsilon_x(y)\|^{-z}, \qquad -1 < \Re z < 1, \tag{4.61}$$

is such that $P_0 = P'$ and $\operatorname{ord} P_z = z - (d+2)$.

By (4.58) the map $f \to f(x, \sigma(x,D))$ is continuous from $S_{||,\text{comp}}(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ into $\mathcal{L}^{(1,\infty)}$. Hence P' is the limit in $\mathcal{L}^{(1,\infty)}$ as $z \to 0$ of the trace class operators P_z, $-1 < \Re z < 0$. Hence P' lies in $\mathcal{L}_0^{(1,\infty)}$ and $\operatorname{Tr}_\omega P'$ vanishes. Thereby $\operatorname{Tr}_\omega P$ depends only on P and $\operatorname{Tr}_\omega P = \operatorname{Tr}_\omega P_{c_P}$.

However the linear functional

$$\tau(c) = \operatorname{Tr}_\omega P_c, \qquad c \in C_c^\infty(\mathbb{R}^{d+1}). \tag{4.62}$$

is positive. Indeed as $c_{\Delta^{-\frac{d+2}{2}}} = \frac{1}{\Gamma(\frac{d+2}{2})} a_0(x) > 0$ we have

$$\tau(c^2) = \operatorname{Tr}_\omega (c^2 (c_{\Delta^{-\frac{d+2}{2}}})^{-1} \Delta^{-\frac{d+2}{2}}) \geq 0. \tag{4.63}$$

So τ must be a measure. As translations are Heisenberg diffeomorphisms with jacobian 1, this measure is translation invariant and thus proportional to the Lebesgue measure. Hence

$$\operatorname{Tr}_\omega P = \tau(c_P) = c_\omega \int c_P(x) = c_\omega \operatorname{Res} P, \qquad \operatorname{ord} P = -(d+2). \tag{4.64}$$

It follows that Tr_ω is proportional to Res on each connected component of M. By (4.54) the constant of proportionality is always equal to $(d+2)^{-1}$. Thus each $P \in \Psi_{\mathcal{V}}^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})$ of order $\leq -(d+2)$ is measurable and its Dixmier trace is equal to $(d+2)^{-1}$ times its non-commutative residue. ■

4.3 Traces and sum of commutators on the $\Psi_{\mathcal{V}}DO$ algebra

Let $(M^{d+1},\mathcal{V})$ be a compact Heisenberg manifold and $\mathcal{E}$ a vector bundle over M. By proposition 4.9 the non-commutative residue is a trace on $\Psi_{\mathcal{V}}^{\mathbb{Z}}(M,\mathcal{E})$ vanishing on $\Psi^{-\infty}(M,\mathcal{E})$. In this section we shall show it is essentially the only one. This is proved independently in [EMM] using the homological techniques of [BrGe]. Here we give an elementary proof based on the ideas of [FGLS].

First we need some lemmas giving criteria for a $\Psi_{\mathcal{V}}DO$ on an open subset U of $\mathbb{R}^{d+1}$ with a $\mathcal{V}$-frame to be a sum of commutators.

Lemma 4.12 *Any $P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(U)$ whose symbol is a sum of ξ-derivatives is a sum of commutators up to a smoothing operator.*

Proof. Let $f \in S^{\mathbb{Z}}(U\times\mathbb{R}^{d+1})$ and set $q(x,\xi) = f(x,\sigma(x,\xi))$. As $[q(x,D),x_j] = \partial_{\xi_j}q(x,D)$ the Heisenberg symbol of $[f(x,\sigma(x,D)),x_j]$ is equal to

$$x_j\#f - f\#x_j = \sum_{k=0}^{d} \partial_{\xi_j}\sigma_k\partial_{\xi_k}f. \tag{4.65}$$

Let $c(x) = (c_{ij}(x))$ be the inverse matrix of the linear map $\sigma(x,.)$ and set $f_{ij}(x,\xi) = c_{ij}(x)f(x,\xi)$. Since the symbol of $\sum_{j=0}^{d}[x_j, f_{ij}(x,\sigma(x,D))]$ is equal to

$$\sum_{j,k=0}^{d} c_{ij}(x)\partial_{\xi_j}\sigma_k(x,\xi)\partial_{\xi_k}f(x,\xi) = \partial_{\xi_i}f(x,\xi), \tag{4.66}$$

we see that $\partial_{\xi_i}f(x,D)$ is a sum of commutators. The lemma follows at once. It follows that any $P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(U)$ whose symbol is a sum of ξ-derivatives is a sum of commutators up to a smoothing operator. ■

Lemma 4.13 *Let $P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(U)$ with a zero symbol in degree $-(d+2)$. Then P is a sum of commutators up to a smoothing operator.*

Proof. Let $f \in S_m(U \times \mathbb{R}^{d+1})$, $m \neq -(d+2)$. As $f(x,\xi)$ is homogeneous in ξ we have

$$2\xi_0 \partial_{\xi_0} f(x,\xi) + \sum_{j=1}^{d} \xi_j \partial_{\xi_j} f(x,\xi) = m f(x,\xi). \tag{4.67}$$

Hence

$$f = \frac{1}{m+d+2}(2\partial_{\xi_0}(\xi_0 f) + \sum_{j=1}^{d} \partial_{\xi_j}(\xi_j f)). \tag{4.68}$$

Now, let $f \sim \sum f_{m-j}$ be the symbol of P. By hypothesis $f_{-(d+2)} = 0$. So for $j = 0, \ldots, d$ there exists $f^{(j)} \in S^*(U \times \mathbb{R}^{d+1})$ such that

$$f^{(j)} \sim \sum_{k \geq 0} \frac{1}{m-k+d+2} \xi_j f_{m-k}. \tag{4.69}$$

Then (4.67) implies

$$f(x,\xi) = 2\partial_{\xi_0} f^{(0)}(x,\xi) + \sum_{j=1}^{d} \partial_{\xi_j} f^{(j)}(x,\xi) \qquad \mod S^{-\infty}(U \times \mathbb{R}^{d+1}). \tag{4.70}$$

Hence by lemma 4.12 the $\Psi_{\mathcal{V}}DO$ operator P is a sum of commutators up to a smoothing one. ■

Lemma 4.14 *Let $P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(U)$ be such that $c_P(x)$ is a sum of derivatives. Then P is a sum of commutators up to a smoothing operator.*

Proof. First suppose $c_P = 0$. Then we have $P = P_0 + P_1$ where P_1 has no symbol of degree $-(d+2)$ and P_0 is given by a kernel of the form

$$k_{P_0}(x,y) = |\varepsilon_x'| a(x, -\varepsilon_x(y)), \tag{4.71}$$

with $a_0 \in \mathcal{K}_0(U \times \mathbb{R}^{d+1})$ homogeneous of degree 0 in the last variable. By lemma 4.13 we know that P_1 is a sum of of commutators up to a smoothing operator. Set

$$a_0(x,y) = \frac{y_0}{\|y\|^4} a(x,y), \qquad a_j(x,y) \frac{y_j^3}{\|y\|^4} a(x,y), \quad 1 \leq j \leq d. \tag{4.72}$$

Denote by $c(x) = (c_{jk}(x))$ the matrix of ε_x', so that $\varepsilon_x(y) = c(x)(x-y)$, and let P_{jk} be the $\Psi_{\mathcal{V}}DO$ with kernel

$$k_{jk}(x,y) = |\varepsilon_x'| c_{jk}(x) a_j(x, -\varepsilon_x(y)). \tag{4.73}$$

Then $\sum[x_k, P_{jk}]$ has kernel

$$|\varepsilon'_x| \sum_{j,k=0}^{d} (x_k - y_k) c_{jk}(x) a_j(x, -\varepsilon_x(y)) = |\varepsilon'_x| a(x, -\varepsilon_x(y)). \tag{4.74}$$

Thus P_0, and *a fortiori* P, is a sum of commutators up to smoothing operators.

Suppose now that $c_P = \sum \partial_{x_j} c_j$ and let P_j be the $\Psi_{\mathcal{V}}DO$ with kernel

$$k_j(x,y) = -c_j(x) \log \|\varepsilon_x(y)\|. \tag{4.75}$$

As $c_{[\partial_{x_j}, P_j]} = \partial_{x_j} c_j$ we have $P = \sum_{j=0}^{d} [\partial_{x_j}, P_j] + R$ with $c_R = 0$. Thus by the first part of the proof P is a sum of commutators up to a smoothing operator. ■

Theorem 4.15 *Let $(M^{d+1}, \mathcal{V})$ be a compact Heisenberg manifold and $\mathcal{E}$ be a vector bundle over M. If M is connected each trace on $\Psi_{\mathcal{V}}^{\mathbb{Z}}(M, \mathcal{E})/\Psi^{-\infty}(M, \mathcal{E})$ is proportional to the non-commutative residue.*

Proof. Let τ be a trace on $\Psi_{\mathcal{V}}^{\mathbb{Z}}(M, \mathcal{E})/\Psi^{-\infty}(M, \mathcal{E})$. We shall see it as a trace on $\Psi_{\mathcal{V}}^{\mathbb{Z}}(M, \mathcal{E})$ vanishing on $\Psi^{-\infty}(M, \mathcal{E})$ and we want to show the existence of some constant λ such that

$$\tau(P) = \lambda \operatorname{Res} P \qquad \forall P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M, \mathcal{E}). \tag{4.76}$$

As M is connected and both sides vanish on smoothing operators, it is enough to prove (4.76) locally for scalar operators on a trivializing Heisenberg chart which is diffeomorphic to $\mathbb{R}^{d+1}$. Then $\mathbb{R}^{d+1}$ is equipped with a $\mathcal{V}$-frame and τ induces a trace on the algebra $\Psi_{\mathcal{V},\text{comp}}^{\mathbb{Z}}(\mathbb{R}^{d+1})$ vanishing on $\Psi_{\text{comp}}^{-\infty}(\mathbb{R}^{d+1})$.

Now if $P \in \Psi_{\mathcal{V},\text{comp}}^{\mathbb{Z}}(\mathbb{R}^{d+1})$ then we have $P = P_{c_P} + Q$, where P_{c_P} is the $\Psi_{\mathcal{V}}DO$ with kernel $-c_P(x) \log \|\varepsilon_x(y)\|$ and $Q \in \Psi_{\mathcal{V},\text{comp}}^{\mathbb{Z}}(\mathbb{R}^{d+1})$ such that $c_Q = 0$. By lemma 4.14 the $\Psi_{\mathcal{V}}DO$ operator Q is a sum of commutators up to a smoothing one. Actually it follows from the proof of the previous lemmas that we can build this construct by means of compactly supported $\Psi_{\mathcal{V}}DO$ operators. Then $\tau(Q)$ vanishes and we have $\tau(P) = \tau(P_c)$.

However, again by lemma 4.14, the functional $c \to \tau(P_c)$ on $C_c^\infty(\mathbb{R}^{d+1})$ vanishes on sum of derivatives. Therefore it is proportional to the Lebesgue measure and there is a constant λ such that

$$\tau(P) = \tau(P_{c_P}) = \lambda \int c_P(x) = \lambda \operatorname{Res} P, \qquad P \in \Psi_{\mathcal{V},\text{comp}}^{\mathbb{Z}}(\mathbb{R}^{d+1}). \tag{4.77}$$

Going back to M we conclude that τ as a trace on $\Psi_{\mathcal{V}}^{\mathbb{Z}}(M, \mathcal{E})$ is proportional to the non-commutative residue. ■

As a smoothing operator is a sum of smoothing commutators if, and only if, its trace vanishes (see [Gu2, appendix]) we get the following corallary.

Corollary 4.16 *Let* $(M^{d+1}, \mathcal{V})$ *be a compact Heisenberg manifold and* $\mathcal{E}$ *be a vector bundle over* M*. Then* $P \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M, \mathcal{E})$ *is a sum of commutators if, and only if, it is of the form*

$$P = Q + R, \tag{4.78}$$

with $Q \in \Psi_{\mathcal{V}}^{\mathbb{Z}}(M, \mathcal{E})$ *and* $R \in \Psi^{-\infty}(M, \mathcal{E})$ *such that*

$$\operatorname{Res} Q = \operatorname{Trace} R = 0. \tag{4.79}$$

Chapter 5

Spectral geometry of Heisenberg and pseudohermitian manifolds

In this last chapter we give geometric applications of the non-commutative residue and the TR-trace. In the first section we define the zeta function of an elliptic sublaplacian and, in the selfadjoint case, we relate its residues and regular values to the coefficients of the heat kernel asymptotics (theorems 5.3 and 5.5).

In section 5.2 we derive variational formulae for zeta functions with respect to C^1 families of sublaplacians. We use them in section 5.3 to produce conformal invariants associate to sublaplacians (theorem 5.14) extending then the results of N.K. Stanton [St].

In section 5.4 we look at the non-commutative geometry of pseudohermitian manifolds. In particular we are able to define the area of a compact three dimensional pseudohermitian manifold and to compute it by an explicit local formula involving the Tanaka-Webster scalar curvature (theorem 5.20).

In the last section we study the index of a square root of an elliptic sublaplacian. First we show that in even dimension the index is always zero and in odd dimension the index is given by the right coefficient of the heat kernel asymptotics (theorem 5.21).

Next using cyclic cohomology and the local index formula of Connes-Moscovici we are able to show the existence of an even homology class whose pairing with the Chern character of a vector bundle gives the index with coefficients in the bundle and we give a local formula for currents represented the components of this homology class as a universal finite linear combination of non-commutative residues (theorem 5.27).

5.1 The zeta function of an elliptic sublaplacian

Let $(M^{d+1}, \mathcal{V})$ be a compact Heisenberg manifold and let Δ be an elliptic sublaplacian on M. We assume here that Δ is either invertible or selfadjoint. Then we can construct its complex powers as in chapter 3 and define the *zeta function* of Δ as the meromorphic function

$$\zeta(s) = \mathrm{TR}\, \Delta^{-s}, \qquad s \in \mathbb{C}. \tag{5.1}$$

Since the non-commutative residue of a differential operator is zero theorem 4.5 gives

Proposition 5.1 *Let $\Sigma = \{\frac{1}{2}k\,;\, k = 0, 1, \ldots, d+2\} \cup (-\frac{1}{2} + \mathbb{Z}_-)$. Then the zeta function (5.1) is holomorphic on $\mathbb{C} \setminus \Sigma$ and has at worst simple pole singularities on Σ with residues*

$$\mathrm{res}_{s=s_0}\, \zeta(s) = 2\, \mathrm{Res}\, \Delta^{-s_0} = 2 \int_M c_{\Delta^{-s_0}}(x), \qquad s_0 \in \Sigma. \tag{5.2}$$

Suppose now that Δ is selfadjoint and let us relate the residues and the regular values of the zeta function to the heat kernel asymptotics of Δ for t small,

$$k_t(x,x) \sim t^{-\frac{d+2}{2}} \sum_{j \geq 0} t^j a_j(\Delta)(x), \tag{5.3}$$

where the $a_j(\Delta)(x)$'s are smooth densities on M.

The idea here is to introduce an auxiliary meromorphic function, directly related to the heat kernel, and which is very much like the zeta function at integer points. This function is

$$\vartheta(s) = \mathrm{TR}\, D_{-s}, \qquad D_{-s} = \frac{1}{\Gamma(s)} \int_0^1 t^s e^{-t\Delta} \frac{dt}{t}. \tag{5.4}$$

Lemma 5.2 *The family (D_{-s}) given by (5.4) is holomorphic on the right half-plane $\{\Re s > 0\}$. It extends to a holomorphic family on the whole $\mathbb{C}$ such that the family*

$$R_s = \Delta^{-s} - D_{-s}, \qquad s \in \mathbb{C}, \tag{5.5}$$

is a holomorphic family of smoothing operators satisfying

$$R_0 = -\Pi_0, \qquad R_{-k} = 0, \qquad k \text{ integer } > 0. \tag{5.6}$$

Proof. That (D_{-s}) is a holomorphic family of $\Psi_{\mathcal{V}}DO$'s on the right half-plane follows from the proof of theorem 3.17. Moreover an integration by parts yields

$$\begin{aligned} \Delta D_{-s} &= \frac{1}{\Gamma(s)} \int_0^1 t^s \Delta e^{-t\Delta} \frac{dt}{t} = \frac{-1}{\Gamma(s)} \int_0^1 t^s \frac{d}{dt}(e^{-t\Delta}) \frac{dt}{t}. \\ &= \frac{-1}{\Gamma(s)} e^{-\Delta} + \frac{s-1}{\Gamma(s)} \int_0^1 t^{s-1} e^{-t\Delta} \frac{dt}{t}. \end{aligned} \tag{5.7}$$

Since $\Gamma(s) = (s-1)\Gamma(s-1)$ we obtain the equality

$$D_{-(s-1)} = \Delta D_{-s} + \Gamma(s)^{-1}e^{-\Delta}. \tag{5.8}$$

As $\Gamma(s)^{-1}$ is a holomorphic function on $\mathbb{C}$ it follows that the family (D_{-s}) extends to a holomorphic family on the whole complex plane.

Now let Π_+ (resp. Π_-) be the orthogonal projection onto the span of the eigenvectors with non-negative (resp. non-positive) eigenvalues. Then the families

$$\Pi_- D_{-s}, \qquad \Pi_-\Delta^{-s}, \qquad \Pi_0 D_{-s} = \frac{1}{s\Gamma(s)}\Pi_0, \qquad \Pi_0\Delta_{-s} = 0, \tag{5.9}$$

are all holomorphic family of smoothing operators over $\mathbb{C}$. Moreover the Mellin formula gives

$$\Pi_+\Delta^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^s\Pi_+ e^{-t\Delta}\frac{dt}{t} = \Pi_+ D_{-s} + \frac{1}{\Gamma(s)}\int_1^\infty t^s e^{-t\Delta}\frac{dt}{t}. \tag{5.10}$$

So using the equalities

$$\int_1^\infty t^s e^{-t\Delta}\frac{dt}{t} = \Pi_+ e^{-\Delta/4}\int_{\frac{1}{2}}^\infty t^s e^{-t\Delta}\frac{dt}{t}e^{-\Delta/4}, \tag{5.11}$$

$$\Delta^{-s} - D_{-s} = \Pi_-(\Delta^{-s} - D_{-s}) + \frac{1}{s\Gamma(s)}\Pi_0 + \frac{1}{\Gamma(s)}\int_1^\infty t^s e^{-t\Delta}\frac{dt}{t}, \tag{5.12}$$

we see that $R_s = \Delta^{-s} - D_{-s}$ is a holomorphic family of smoothing operators on $\mathbb{C}$.

Finally as $\Gamma(s)$ has simple poles at negative integers with residue 1 at zero, the equality (5.12) shows that $R_0 = -\Pi_0$ and $R_{-k} = 0$ if k is a non-negative integer. ■

Theorem 5.3 *Suppose $d+1$ odd, $d+1 = 2n+1$, and assume that Δ is selfadjoint. Then:*

1) For $k = 1, \ldots, n+1$ we have

$$\operatorname{res}_{s=k}\zeta(s) = \frac{1}{2}\operatorname{Res}\Delta^{-k} = \frac{1}{(k-1)!}\int_M a_{n+1-k}(\Delta)(x). \tag{5.13}$$

2) At $s = 0$ the regular value is

$$\zeta(0) = \int_M a_{n+1}(\Delta)(x) - \dim\ker\Delta. \tag{5.14}$$

3) For any non-positive integer $-k$ we have

$$\zeta(-k) = (-1)^{k-1}(k-1)! \int_M a_{n+1+k}(\Delta)(x). \tag{5.15}$$

Proof. By lemma 5.2 the family $R_s = \Delta^{-s} - D_{-s}$ is a holomorphic family of smoothing operators on $\mathbb{C}$. So by remark 4.7 we need only to look at the auxiliary function

$$\vartheta(s) = \mathrm{TR}\, D_{-s}, \qquad s \in \mathbb{C}. \tag{5.16}$$

Moreover the asymptotic (5.3) implies that for any integer N we have

$$k_t(x,x) = t^{-\frac{d+2}{2}} \sum_{j=0}^{N} t^j a_j(\Delta)(x) + t^{N-(n+1)} r_N(x,t), \tag{5.17}$$

with $r_N(x,t)$ bounded for t small. Thus

$$\int_0^1 t^s k_t(x,x) \frac{dt}{t} = \sum_{j=0}^{N} \frac{1}{s+j-(n+1)} a_j(\Delta)(x) + h_{N,s}(x), \tag{5.18}$$

where $h_{N,s}(x)$ is a holomorphic family of densities for $\Re s > n+1-N$. Integrating over M we obtain

$$\mathrm{Trace}\, D_{-s} = \frac{1}{\Gamma(s)} \sum_{j=0}^{N} \frac{1}{s+j-(n+1)} \int_M a_j(\Delta)(x) + \frac{1}{\Gamma(s)} h_N(s), \tag{5.19}$$

where $h_N(s)$ is a holomorphic function for $\Re s > n+1-N$. The conclusion follows from this last equality and the properties of the Gamma function, noting that $\mathrm{Trace}\, R_s$ is equal to $-\dim\ker\Delta$ at $s=0$ and vanishes at non-positive integers. ■

Remark 5.4 The above computations are local and (5.18) shows that we can calculate explicitly the densities $c_{\Delta^{-k}}(x)$ and the regular values of $t_{\Delta^{-s}}$ at negative integers. We get

$$c_{\Delta^{-k}}(x) = \frac{1}{(k-1)!} a_{n+1-k}(\Delta)(x), \qquad k = 1, \ldots, n+1, \tag{5.20}$$

$$t_{\Delta^k}(x) = (-1)^{k-1}(k-1)! a_{n+1+k}(\Delta)(x), \qquad k \in \mathbb{N}. \tag{5.21}$$

Arguing similarly in the even dimensional case we obtain:

Theorem 5.5 *Suppose $d+1 = 2n$ even and assume Δ selfadjoint. Then:*

1) For $k = -n, -n+1, \ldots$ we have

$$\operatorname{res}_{s=\frac{1}{2}-k} = \operatorname{Res}\Delta^{-\frac{1}{2}+k} = \frac{1}{\Gamma(\frac{1}{2}-k)}\int_M a_{n+k}(x). \tag{5.22}$$

2) The regular value at $s = 0$ is given by

$$\zeta(0) = -\dim\ker\Delta. \tag{5.23}$$

3) The regular values at non-positive integers all vanish.

5.2 Variational formulae and homotopy invariance

In this section we shall derive variational formulae for the zeta functions associate to a C^1-family of elliptic sublaplacians on a compact Heisenberg manifold. The first step is to justify the switching over of the trace and the derivation with respect of the parameter. This follows from a more general variational formula for the TR-trace which is an almost immediate consequence of the procedure carried out to construct it.

Before achieving that let us first define C^1-family of $\Psi_{\mathcal{V}}DO$ operators over an open interval I of $\mathbb{R}$.

Definition 5.6 *For $m \in \mathbb{C}$ a family $(f_\epsilon)_{\epsilon\in I}$ with values $S^m(\mathbb{R}^{d+1})$ is C^1 if:*

(i) for ξ fixed $f_\epsilon(\xi)$ is C^1 function of ϵ;

(ii) for any j the homogeneous symbols $f_{\epsilon,m-j}$ of degree $m-j$ of f_ϵ depends in a C^1 way on ϵ;

(iii) the bounds of the asymptotic $f_\epsilon \sim \sum f_{\epsilon,m-j}$ are uniform with respect to the C^1-topology.

If U is an open subset of $\mathbb{R}^{d+1}$ we can as well define C^1-families of symbols of order m on $U \times \mathbb{R}^{d+1}$, obtaining $C^1\hat{\otimes}C^\infty$-families of symbols of order m on $\mathbb{R}^{d+1}$.

If U is equipped with a hyperplane bundle $\mathcal{V} \subset TU$ and a $\mathcal{V}$-frame, we define a C^1-family of $\Psi_{\mathcal{V}}DO$ operators on U as a family (P_ϵ) of $\Psi_{\mathcal{V}}DO$'s of the form

$$P_\epsilon = f_\epsilon(x, \sigma(x, D)) + R_\epsilon, \tag{5.24}$$

with (f_ϵ) a C^1-family of symbols on $U \times \mathbb{R}^{d+1}$ and (R_ϵ) a C^1-family of smoothing operators.

The notion of C^1-family of $\Psi_{\mathcal{V}}DO$'s is stable under the composition of operators and invariant by Heisenberg diffeomorphisms. So we can define them on any Heisenberg manifold. Moreover if Δ_ϵ is a C^1-family of sublaplacians then the parametrix construction of [BG] can be carried out so that to obtain a C^1-family of parametrices.

If $\Omega \subset \mathbb{C}$ is an open and $\Lambda \subset \mathbb{C}\setminus 0$ is a pseudocone we can similarly define $C^1(I)\hat{\otimes}\,\mathrm{Hol}(\Omega)$-families and $C^1(I)\hat{\otimes}\,\mathrm{Hol}^p(\Lambda)$-families of $\Psi_{\mathcal{V}}DO$ operators using the same procedures as in chapters 2 and 3.

The notion of $C^1\hat{\otimes}\,\mathrm{Hol}$-family is very relevant for our purpose. Remember that theorem 4.5 followed from lemma 4.4 which was itself a straightforward extension to smooth families of symbols of lemma 4.2. In the same way we have an extension of lemma 4.2 to $C^1\hat{\otimes}C^\infty$-families.

Lemma 5.7 *Let $(f_{\epsilon,z})$ be a $C^1(I)\hat{\otimes}\,\mathrm{Hol}(\Omega)$-family of symbols on $U\times\mathbb{R}^{d+1}$.*

1) If $\mathrm{ord} f_{\epsilon,z}\notin\mathbb{Z}$ then $\tilde{L}(f_{\epsilon,s}(x,.))$ is holomorphic from Ω into $C^1(I)\hat{\otimes}C^\infty(U)$ and we have

$$\partial_\epsilon\tilde{L}(f_{\epsilon,s}(x,.)) = \tilde{L}(\partial_\epsilon f_{\epsilon,s}(x,.)). \tag{5.25}$$

2) If $\mathrm{ord} f_{\epsilon,z}=z$ around some integer m then $\tilde{L}(f_{\epsilon,s}(x,.))$ has a simple pole singularity near $z=m$ and the equality (5.25) continues to hold as an equality of meromorphic function near $z=m$.

Combining this lemma with the proof of theorem 4.5 we obtain:

Proposition 5.8 *Let $(M,\mathcal{V})$ be a compact Heisenberg manifold and let $(P_{\epsilon,s})$ be a $C^1\hat{\otimes}\,\mathrm{Hol}$-family of $\Psi_{\mathcal{V}}DO$ operators on M.*

1) If $\mathrm{ord}P_{\epsilon,s}\notin\mathbb{Z}$ then $\mathrm{TR}\,P_{\epsilon,s}$ is holomorphic for the C^1-topology and we have

$$\partial_\epsilon\,\mathrm{TR}\,P_{\epsilon,s} = \mathrm{TR}\,\partial_\epsilon P_{\epsilon,s} \tag{5.26}$$

2) Suppose that $\mathrm{ord}P_{\epsilon,s}=z$ around some integer m then $\mathrm{TR}\,P_{\epsilon,s}$ has a simple pole singularity for the C^1-topology near $z=m$ and (5.26) holds as an equality of meromorphic functions near $z=m$.

Proposition 5.9 *Let $(M^{d+1},\mathcal{V})$ be a compact Heisenberg manifold and let (Δ_ϵ) be a C^1-family of elliptic sublaplacians on M. We make the following assumptions:*

(i) the operator Δ_ϵ is either invertible or selfadjoint;

(ii) there exists a connected open pseudocone $\Lambda\subset\mathbb{C}\setminus 0$ such that $\Lambda\cap\mathrm{sp}\,\Delta_\epsilon=\emptyset$ for any ϵ.

We can the define the complex powers for Δ_ϵ as in chapter 3 by means of an integration contour contained in Λ. Then the function $\mathrm{TR}\,\Delta_\epsilon^{-s}$ *is meromorphic for the C^1-topology and we have*

$$\partial_\epsilon \,\mathrm{TR}\,\Delta_\epsilon^{-s} = -s\,\mathrm{TR}\,\partial_\epsilon(\Delta_\epsilon)\Delta_\epsilon^{-s-1}. \tag{5.27}$$

In particular,

$$\partial_\epsilon \,\mathrm{Res}\,\Delta_\epsilon^{-k} = -k\,\mathrm{Res}\,\partial_\epsilon(\Delta_\epsilon)\Delta_\epsilon^{-k-1} \qquad k = 0, \frac{1}{2}, \ldots, \frac{d+2}{2}. \tag{5.28}$$

Proof. The construction of an asymptotic resolvent for $\Delta_\epsilon - \lambda$, $\lambda \in \Lambda$, in chapter 2 can be carried out smoothly with respect to the parameter ϵ. As the domain of Δ_ϵ doesn't depend on ϵ and for any ϵ there is no spectrum of Δ_ϵ in Λ, it follows that $(\Delta_\epsilon - \lambda)^{-1}$ is a C^1-family of parametric $\Psi_\mathcal{V}DO$ operators. Then the construction of the complex powers of Δ_ϵ of section 3.3 gives a $C^1\hat{\otimes}\,\mathrm{Hol}$-family of $\Psi_\mathcal{V}DO$ operators. So by proposition 5.8 the function $\mathrm{TR}\,\Delta_\epsilon^{-s}$ is meromorphic for the C^1-topology and we have the equality of meromorphic functions

$$\partial_\epsilon \,\mathrm{TR}\,\Delta_\epsilon^{-s} = \mathrm{TR}\,\partial_\epsilon(\Delta_\epsilon^{-s}). \tag{5.29}$$

Therefore it is enough to show that for $\Re s >> 0$ we have

$$\mathrm{TR}\,\partial_\epsilon(\Delta_\epsilon^{-s}) = -s\,\mathrm{TR}\,\partial_\epsilon\Delta_\epsilon\,\Delta_\epsilon^{-s} \tag{5.30}$$

Now let m be an integer $> -\frac{d+2}{2}$. Then (Δ_ϵ^{-m}) is a C^1-family of trace class operators and for $\Re s < 0$ we have

$$\begin{aligned} \partial_\epsilon \,\mathrm{Trace}\,\Delta_\epsilon^{-m+s} &= \frac{i}{2\pi}\int_\Gamma \lambda^s \partial_\epsilon \,\mathrm{Trace}\,\Delta_\epsilon^{-m}(\Delta_\epsilon - \lambda)^{-1} d\lambda, \\ &= -m\,\mathrm{Trace}\,\partial_\epsilon\Delta_\epsilon\Delta_\epsilon^{-m-1}\frac{i}{2\pi}\int_\Gamma \lambda^s(\Delta_\epsilon-\lambda)^{-1}d\lambda \\ &- \mathrm{Trace}\,\partial_\epsilon\Delta_\epsilon\Delta_\epsilon^{-m-1}\frac{i}{2\pi}\int_\Gamma \lambda^s(\Delta_\epsilon-\lambda)^{-2}d\lambda. \end{aligned} \tag{5.31}$$

As an integration by parts yields

$$\frac{i}{2\pi}\int_\Gamma \lambda^s(\Delta_\epsilon-\lambda)^{-2}d\lambda = \frac{-is}{2\pi}\int_\Gamma \lambda^{s-1}(\Delta_\epsilon-\lambda)^{-1}d\lambda = -s\Delta_\epsilon^{-s-1}, \tag{5.32}$$

we conclude that

$$\partial_\epsilon \,\mathrm{Trace}\,\Delta_\epsilon^{-m+s} = (s-m)\,\mathrm{Trace}\,\partial_\epsilon\Delta_\epsilon\Delta_\epsilon^{-m+s-1} \qquad \Re s < 0. \tag{5.33}$$

Hence (5.27) holds for $\Re s > m$. As both sides of this equality are meromorphic functions, it holds on the whole $\mathbb{C}$. ■

Corollary 5.10 *Let Δ_0 and Δ_1 be two elliptic sublaplacians on M which can be connected to each other by means of a C^1-family satisfying the assumptions of proposition 5.9. Then the regular values at zero of the zeta functions associated to Δ_0 and Δ_1 coincide.*

5.3 Conformal invariants of sublaplacians

Let (M^n, g) be a compact Riemannian manifold and let $\Box_g$ be the conformal Laplacian on M,

$$\Box_g = d^*d + \frac{1}{4}\frac{n-2}{n-1}s_n, \tag{5.34}$$

where s_n is the scalar curvature. This operator, also called Yamabe operator, was studied in order to solve the Yamabe problem of finding within the conformal class of g a metric with constant scalar curvature (see [Au] and [Sc]). It transforms conformally under a conformal change $g \to e^{2f}g$ of metrics, i.e.

$$\Box_{e^{2f}g} = e^{-(\frac{n}{2}+1)f}\Box_g e^{(\frac{n}{2}-1)f}, \qquad f \in C^\infty(M). \tag{5.35}$$

For t small the kernel of $e^{-t\Box_g}$ admits on the diagonal an asymptotics of the form

$$k_t(x,x) \sim t^{-\frac{n}{2}} \sum t^k a_k(\Box_g)(x), \tag{5.36}$$

where the $a_k(\Box_g)(x)$'s are densities on M given by local invariants in the jets of the metric [Gi].

If $F(g)$ is a function of the metric g, or of the contact form θ below, we set

$$\delta_f = \frac{\partial}{\partial \epsilon} F(e^{2\epsilon f}g)_{|\epsilon=0}, \qquad f \in C^\infty(M). \tag{5.37}$$

In [BØ1] and [PR] the following theorem is proved.

Theorem 5.11 ([BØ1], [PR]) *Let (M^{2n}, g) be an even dimensional compact Riemannian manifold and let $f \in C^\infty(M)$. Then:*

1) We have

$$a_{n-1}(\Box_{e^{2f}g})(x) = e^{2f(x)}a_{n-1}(\Box_g)(x), \tag{5.38}$$

i.e. $a_{n-1}(\Box_g, x)$ is a local conformal invariant of weight -2.

2) For any integer k,

$$\delta_f \int_M a_k(\Box_g)(x) = 2(n-k)\int_M f(x)a_k(\Box_g)(x), \tag{5.39}$$

In particular $A_n = \int_M a_n(\Box_g, x)$ is a conformal invariant.

There is an analogous result for pseudohermitian manifolds proved by N.K. Stanton [St]. Let (M^{2n+1}, θ) be a compact pseudohermitian manifold and let $\boxdot_\theta$ be the conformal sublaplacian,

$$\boxdot_\theta = \Delta_b + \frac{n}{n+2} R_n, \tag{5.40}$$

where R_n is the Tanaka-Webster scalar curvature. The operator was introduced in [Le] in order to solve the corresponding Yamabe problem on CR manifolds [JL1]. Under a conformal change of contact form $\theta \to e^{2f}\theta$ the operator $\boxdot_\theta$ transforms into

$$\boxdot_{e^{2f}\theta} = e^{-(n+2)f} \, \boxdot_\theta \, e^{nf}. \tag{5.41}$$

By proposition 1.53 the heat kernel of $\boxdot_\theta$ admits on the diagonal an asymptotics for t small in the form

$$k_t(x,x) \sim t^{-(n+1)} \sum_{j \geq 0} t^j a_j(\boxdot_\theta)(x). \tag{5.42}$$

Then N.K. Stanton proved:

Theorem 5.12 ([St]) *Let (M^{2n+1}, θ) be a compact pseudohermitian manifold. Then*

1) For any $f \in C^\infty(M)$ we have

$$a_n(\boxdot_{e^{2f}\theta})(x) = e^{2f(x)} a_n(\boxdot_\theta)(x). \tag{5.43}$$

In other words $a_n(\boxdot_{e^{2f}\theta})(x)$ is a local conformal invariant of weight -2.

2) We define a global conformal invariant by setting

$$A_{n+1} = \int_M a_{n+1}(\boxdot_{e^{2f}\theta})(x). \tag{5.44}$$

We shall give here a shorter proof of the first assertion and goes a little bit further with the second one. To this end we need the following lemma.

Lemma 5.13 *Let (M^{2n+1}, θ) be a compact pseudohermitian manifold. For $f \in C^\infty(M)$ define the C^1-family of sublaplacians*

$$\boxdot_{\theta,\epsilon} = \boxdot_{e^{2\epsilon f}\theta}, \qquad -1 < \epsilon < 2. \tag{5.45}$$

Then the family $(\boxdot_{\theta,\epsilon})$ satisfies to the assumptions of the proposition 5.9.

Proof. Let $u_0, \ldots, u_l$ be an orthonormal basis of $\ker \Box_\theta$. The kernel of the projection Π_θ onto $\ker \Box_\theta$ is then given by

$$\pi_\theta(x,y) = \sum_{j=0}^{l} u_j(x)\bar{u}_j(y)(d\theta)^n \wedge \theta(y). \tag{5.46}$$

As $e^{-n\epsilon f}u_0, \ldots, e^{-n\epsilon f}u_l$ is a basis for $\ker \Delta_\epsilon$ a Gram-Schmidt orthonormalisation produces an orthonormal basis $v_{0,\epsilon}, \ldots, v_{l,\epsilon}$, with respect the inner-product induced by the volume form $e^{2(n+2)\epsilon f}(d\theta)^n \wedge \theta$, such that $v_{j,\epsilon}$ is C^1 with respect to ϵ. Then it follows from (5.46) that the orthogonal projection Π_ϵ onto $\ker \Box_{\theta,\epsilon}$ is a C^1-family of smoothing operators. Then the equality

$$\Box_{\theta,\epsilon}^{-1} = (\Box_{\theta,\epsilon} + \Pi_\epsilon)^{-1} - \Pi_\epsilon, \tag{5.47}$$

shows that $\Box_{\theta,\epsilon}^{-1}$ is a C^1-family of bounded operators and there exists $C > 0$ such that

$$\| \Box_{\theta,\epsilon}^{-1} \| \leq C, \qquad -1 \leq \epsilon \leq 2. \tag{5.48}$$

This implies that $\Box_{\theta,\epsilon}$ has no eigenvalue, except maybe 0, in the interval $]-C^{-1}, C^{-1}[$. Hence the family of sublaplacians $\Box_{\theta,\epsilon}^{-1}$ satisfies to the assumptions of proposition 5.9. ■

Theorem 5.14 *Let (M^{2n+1}, θ) be a compact pseudohermitian manifold and let $f \in C^\infty(M)$.*

1) We have

$$a_n(\Box_{e^{2f}\theta})(x) = e^{2f(x)}a_n(\Box_\theta)(x), \tag{5.49}$$

i.e. $a_n(\Box_{e^{2f}\theta})(x)$ is a local conformal invariant of weight -2.

2) We have the equality of meromorphic functions

$$\delta_f \operatorname{TR} \Box_\theta^{-s} = 2s \operatorname{TR} f\, \Box_\theta^{-s}. \tag{5.50}$$

Hence $\zeta_{\Box_\theta}(0)$ is a conformal invariant.

3) For any integer k,

$$\delta_f \int_M a_k(\Box_\theta)(x) = 2(n+1-k)\int_M f(x)a_k(\Box_\theta)(x). \tag{5.51}$$

Thus $A_{n+1} = \int_M a_{n+1}(\Box_{e^{2f}\theta})(x)$ is a conformal invariant.

Proof. Let Π_0 be the orthogonal projection onto $\ker \boxdot_\theta$. Then

$$e^{-nf}\, \boxdot_\theta^{-1}\, e^{(n+2)f} \boxdot_{e^{2f}\theta} = e^{-nf}\, \boxdot_\theta^{-1}\, \boxdot_\theta e^{nf} = 1 - e^{-nf}\Pi_0 e^{nf}. \tag{5.52}$$

So $e^{-nf}\, \boxdot_\theta^{-1}\, e^{(n+2)f}$ is a parametrix for $\boxdot_{e^{2f}\theta}$ and differs from $\boxdot_{e^{2f}\theta}^{-1}$ only by a smoothing operator. Therefore it follows from theorem 5.3 that

$$a_n(\boxdot_\theta)(x) = c_{\boxdot_\theta^{-1}}(x) = c_{e^{-nf}\boxdot_\theta^{-1}e^{(n+2)f}}(x) = e^{2f(x)}c_{\boxdot_\theta^{-1}} = e^{2f(x)}a_n(\boxdot_\theta)(x). \tag{5.53}$$

On the other hand, by lemma 5.13 the family $\boxdot_{\theta,\epsilon} = \boxdot_{e^{2\epsilon f}\theta}$ satisfies to the assumptions of proposition 5.9. So $\operatorname{TR} \boxdot_{\theta,\epsilon}^{-s}$ is meromorphic for the C^1-topology and we have the equality of meromorphic functions

$$\delta_f \operatorname{TR} \boxdot_\theta^{-s} = -s \operatorname{TR}(\partial_\epsilon \boxdot_{\theta,\epsilon})_{\epsilon=0} \boxdot_\theta^{-s-1} = 2s \operatorname{TR} f\, \boxdot_\theta^{-s}\,. \tag{5.54}$$

Hence the second assertion.

Finally the last assertion follows from the second one and theorem 5.3, noting that $\dim \ker \boxdot_\theta$ is a conformal invariant. ■

Remark 5.15 The last assertion answers positively to a conjecture raised by Branson-Ørsted [BØ2].

5.4 Non-commutative geometry of pseudohermitian manifolds

Let (M^{2n+1}, θ) be a compact pseudohermitian manifold. Then proposition 1.55 and theorem 5.3 express the non-commutative residues of the geometric sublaplacians as integrals of universal polynomials in the Tanaka-Webster connection. For instance:

Proposition 5.16 *Let Δ_b be the pseudohermitian sublaplacian on (M, θ). Then*

$$\operatorname{Res} \Delta_b^{-(n+1)} = \alpha_n \int_M (d\theta)^n \wedge \theta, \qquad \operatorname{Res} \Delta_b^{-n} = \beta_n \int_M R_n (d\theta)^n \wedge \theta, \tag{5.55}$$

where α_n and β_n are universal constants and R_n is the Tanaka-Webster scalar curvature.

Remark 5.17 These equalities are pseudohermitian analogues of the corresponding results in Riemannian geometry (see [Co1], [Kast], [KaW]).

By remark 5.4 we actually have

$$c_{\Delta_b^{-(n+1)}}(x) = \alpha_n (d\theta)^n \wedge \theta(x), \qquad c_{\Delta_b^{-n}}(x) = \beta_n R_n (d\theta)^n \wedge \theta(x). \tag{5.56}$$

Combining with theorem 4.11 we obtain:

Corollary 5.18 *For any $f \in C^\infty(M)$ we have*

$$\text{(5.57)} \qquad ⨍ f\Delta_b^{-(n+1)} = \alpha_n \int_M f(d\theta)^n \wedge \theta.$$

So extrapolating [Co4] we can interpret $ds = \alpha_n^{\frac{-1}{2(n+1)}} \Delta_b^{-\frac{1}{2}}$ as a length element and define the area of (M,θ) as follows.

Definition 5.19 *The area of (M,θ) is*

$$\text{(5.58)} \qquad \operatorname{area}_\theta M = \operatorname{Res} ds^2 = \alpha_n^{\frac{-1}{(n+1)}} \operatorname{Res} \Delta_b^{-1}.$$

Theorem 5.20 *For any 3-dimensional pseudohermitian manifold (M^3,θ) we have*

$$\text{(5.59)} \qquad \operatorname{area}_\theta M = \frac{1}{8\sqrt{2}} \int_{M^3} R_1 d\theta \wedge \theta.$$

Proof. By proposition 5.16 there exists a constant β_1 such that for any 3-dimensional pseudohermitian manifold (M^3,θ) we have

$$\text{(5.60)} \qquad \operatorname{area}_\theta M = \frac{\beta_1}{\sqrt{\alpha_1}} \int_M R_3 d\theta \wedge \theta.$$

To compute the ratio $\frac{\beta_1}{\sqrt{\alpha_1}}$ we need only to look at the specific example of the unit sphere S^3 of $\mathbb{C}^2$ with contact form $\theta = \frac{i}{2}(z_1 d\bar{z}_1 + z_2 d\bar{z}_2)$. By [We] the scalar curvature R_1 is then equal to 4 and thus we get

$$\text{(5.61)} \qquad \frac{\beta_1}{\sqrt{\alpha_1}} = \frac{\operatorname{Res}\Delta_b^{-1}}{4\int_{S^3} d\theta\wedge\theta}\Big(\frac{\operatorname{Res}\Delta_b^{-2}}{\int_{S^3} d\theta\wedge\theta}\Big)^{\frac{-1}{2}} = \frac{1}{4}\frac{\operatorname{Res}\Delta_b^{-1}}{\sqrt{\operatorname{Res}\Delta_b^{-2}}}\Big(\int_{S^3} d\theta\wedge\theta\Big)^{\frac{-1}{2}}.$$

On the other hand, by theorem 5.3 as $t \to 0^+$ we have

$$\text{(5.62)} \qquad \operatorname{Trace} e^{-\Delta_b} = \frac{1}{2t^2}\operatorname{Res}\Delta_b^{-2} + \frac{1}{2t}\operatorname{Res}\Delta_b^{-1} + \mathrm{O}(1).$$

Since $R_1 = 4$ here, we have $\Delta_b = \boxdot_\theta - 1$. So using [St, theorem 4.34] we obtain

$$\text{(5.63)} \qquad \operatorname{Trace} e^{-\Delta_b} = e^t\Big(\frac{\pi^2}{16t^2} + \mathrm{O}(t^\infty)\Big) = \frac{\pi^2}{16t^2} + \frac{\pi^2}{16t} + \mathrm{O}(1).$$

Thus $\operatorname{Res}\Delta_b^{-2} = \operatorname{Res}\Delta_b^{-1} = \frac{\pi^2}{8}$ and we get

$$\text{(5.64)} \qquad \frac{\beta_1}{\sqrt{\alpha_1}} = \frac{\pi}{8\sqrt{2}}\Big(\int_{S^3} d\theta\wedge\theta\Big)^{\frac{-1}{2}}.$$

It remains then to compute $\int_{S^3} d\theta \wedge \theta$. We have

$$\begin{aligned} d\theta \wedge \theta &= \frac{-1}{4}(z_2 dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 + z_1 dz_1 \wedge dz_2 \wedge d\bar{z}_2) \\ &= \frac{1}{2}\imath_R(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2), \end{aligned} \tag{5.65}$$

where $R = \frac{\partial}{\partial r}$ is the radial vector fields. Thus

$$\operatorname{vol}_\theta S^3 = \int_{S^3} d\theta \wedge \theta = \frac{1}{2}|S^3| = \pi^2, \tag{5.66}$$

which finally yields $\beta_1 = \frac{1}{8\sqrt{2}}$. ■

5.5 Local index formulae

Let D be an order 1 selfadjoint $\Psi_{\mathcal{V}}DO$ operator on a compact Heisenberg manifold $(M^{d+1}, \mathcal{V})$ acting on a the sections of a vector bundle $\mathcal{S}$ over M. We assume that D is elliptic in the Heisenberg calculus and it anti-commutes with a $\mathbb{Z}_2$-grading γ on $\mathcal{S}$. With respect to this grading we can decompose $\mathcal{S}$ as a direct sum

$$\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-, \tag{5.67}$$

and write D as

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \qquad D_\pm : \mathcal{S}_\pm \to \mathcal{S}_\mp. \tag{5.68}$$

By definition the index of D is

$$\operatorname{ind} D = \operatorname{ind} D^+ = \dim\ker D^+ - \dim\ker D^-, \tag{5.69}$$

The aim of this section is to compute by a local formula the index of D.

Theorem 5.21 *Assume that D^2 is a sublaplacian, so that its heat kernel has an asymptotic on the diagonal of the form*

$$k_t(x,x) \sim t^{-\frac{d+2}{2}} \sum_{j\geq 0} t^j a_j(D^2)(x), \tag{5.70}$$

where the $a_j(D^2)(x)$ are smooth densities on M with values in $\operatorname{END}\mathcal{S}$.

1) *If $d+1$ is even we have* $\operatorname{ind} D = 0$.

2) *If $d+1$ is odd, $d+1 = 2n+1$, then*

$$\operatorname{ind} D = \int_M \operatorname{Str}_{\mathcal{S}} a_0(D^2)(x). \tag{5.71}$$

Proof. We have

$$(5.72)\qquad D^2 = \begin{pmatrix} \Delta^+ & 0 \\ 0 & \Delta^- \end{pmatrix}, \qquad \Delta^\pm = D^\mp D^\pm.$$

The idea of the proof is to express the index of D as the difference of the zeta functions of Δ^+ and Δ^-,

$$(5.73)\qquad \zeta_\pm = \mathrm{TR}(\Delta^\pm)^{-s}.$$

In fact for $\Re s > \frac{1}{2}(d+2)$ we have

$$(5.74)\quad \zeta_+(s) - \zeta_-(s) = \sum_{\lambda>0} \lambda^s(\dim\ker(\Delta^+ - \lambda) - \dim\ker(\Delta^- - \lambda)) = 0,$$

for D induces for any $\lambda > 0$ a bijection between $\ker(\Delta^+ - \lambda)$ and $\ker(\Delta^- - \lambda)$. So if $d+1$ is even then by theorem 5.5 we have

$$(5.75)\qquad 0 = \zeta_+(0) - \zeta_-(0) = -\dim\ker\Delta^+ + \dim\ker\Delta^- = -\operatorname{ind} D,$$

while if $d+1 = 2n+1$ is odd theorem 5.3 gives
(5.76)

$$\operatorname{ind} D = \int_M \operatorname{tr}_{\mathcal{E}^+} a_{(n+1)}(\Delta^+)(x) - \int_M \operatorname{tr}_{\mathcal{E}^-} a_{(n+1)}(\Delta^-)(x) = \int_M \operatorname{Str}_{\mathcal{E}} a_{(n+1)}(D^2)(x),$$

which completes the proof. ■

Remark 5.22 It is actually possible to show that the square D^2 of any elliptic $\Psi_{\mathcal{V}}DO$ operator D of order 1 has an heat kernel asymptotic of the kind of (5.70), adding logarithmic terms as in [DG] and [GruS]. So the theorem holds for any odd selfdajoint elliptic $\Psi_{\mathcal{V}}DO$ of order 1.

Now let $\mathcal{E}$ be a Hermitian vector bundle and let ∇ be a Hermitian connection on $\mathcal{E}$, i.e.

$$(5.77)\qquad \langle\nabla\xi, \eta\rangle - \langle\xi, \nabla\eta\rangle = d\langle\xi, \eta\rangle, \qquad \xi, \eta \in C^\infty(M, \mathcal{E}).$$

We form the twist of D by ∇ as follows. By [Co1, prop. VI.1.4] we define a define a morphism of $C^\infty(M)$-modules $\pi : C^\infty(M, \Lambda T^*M) \to C^\infty(M, \operatorname{End}\mathcal{S})$ by letting

$$(5.78)\qquad \pi(f^0 df^1 \ldots df^n) = f^0[D, f^1]\ldots[D, f^n], \qquad f^j \in C^\infty(M).$$

For example, in the case of the Dirac operator on a spin Riemannian manifold we would have got the Clifford representation. We get then a morphism of $C^\infty(M)$-modules $\pi : C^\infty(M, \mathcal{S} \otimes \Lambda T^*M) \to C^\infty(M, \operatorname{End}\mathcal{S})$. The twisted operator $D_{\nabla,\mathcal{E}}$ is the differential operator acting on $C^\infty(M, \mathcal{S}\otimes\mathcal{E})$ defined by

$$D_{\nabla,\mathcal{E}} = D \otimes 1 + \pi\nabla, \tag{5.79}$$

where $\pi\nabla$ is given by the compositions

$$C^\infty(M, \mathcal{S}\otimes\mathcal{E}) \xrightarrow{1\otimes\nabla} C^\infty(M, \mathcal{S}\otimes\Lambda T^*\otimes\mathcal{E}) \xrightarrow{\pi\otimes 1} C^\infty(M, \mathcal{S}\otimes\mathcal{E}). \tag{5.80}$$

However by Serre-Swan theorem the map $\mathcal{E} \to C^\infty(M, \mathcal{E})$ induces an isomorphism in K-theory,

$$K^0(M) \simeq K_0(\mathcal{A}), \tag{5.81}$$

where $\mathcal{A}$ is the (Fréchet) algebra $C^\infty(M)$. Under this isomorphism the definition of $D_{\nabla,\mathcal{E}}$ coincides the one given in [Mo], so that

$$\operatorname{ind} D_{\nabla,\mathcal{E}} = \operatorname{ind}_D[\mathcal{E}], \tag{5.82}$$

where $[\mathcal{E}]$ is the class of $\mathcal{E}$ in $K^0(M) \simeq K_0(\mathcal{A})$ and ind_D is the index map from $K_0(\mathcal{A})$ into $\mathbb{Z}$ ([At], [Kas], [Co1]).

The cyclic cohomology [Co1] can be presented as follows. First the Hochschild cohomology $H^*(\mathcal{A}, \mathcal{A}^*)$ is the cohomology of the complex of cochains,

$$\begin{aligned} C^n(\mathcal{A}) &= \{\text{continuous } (n+1)\text{-linear form on } \mathcal{A}\} \quad \text{if } n \geq 0, \\ &= 0 \qquad \text{if } n < 0, \end{aligned} \tag{5.83}$$

with coboundary

$$\begin{aligned} b\psi(a^0, \cdots, a^{n+1}) &= \sum (-1)^j \psi(a^0, \cdots, a^j a^{j+1}, \cdots, a^{n+1}) \\ &\quad + (-1)^{n+1}\psi(a^{n+1}a^0, \cdots, a^n) \qquad \forall a^j \in \mathcal{A}. \end{aligned} \tag{5.84}$$

The cyclic cohomology is the cohomology of the sub-complex $(C^*_\lambda(\mathcal{A}), d)$ of the Hochschild complex consisting in cyclic cochains, i.e.

$$\psi(a^1, \cdots, a^n, a^0) = (-1)^n \psi(a^0, \cdots, a^n) \qquad a^j \in \mathcal{A}. \tag{5.85}$$

It can equivalently described in terms of the second filtration of the (b, B)-bicomplex defined as follows. Let

$$C^{n,m}(\mathcal{A}) = C^{n-m}(\mathcal{A}), \qquad n, m \in \mathbb{N}, \tag{5.86}$$

take b as vertical differential and as horizontal differential take $B : C^m(\mathcal{A}) \to C^{m-1}(\mathcal{A})$ given by
(5.87)

$$B = AB_0, \qquad (A\phi)(a^0, \cdots, a^{m-1}) = \sum (-1)^{(m-1)j} \psi(a^j, \cdots, a^{j-1}),$$

$$(5.88)\qquad B_0\psi(a^0,\cdots,a^{m-1})=\psi(1,a^0,\cdots,a^{m-1}),\qquad a^j\in\mathcal{A}.$$

One can check that $b^2=B^2=bB+Bb=0$ so that $(C^{*,*},b,B)$ is really a bicomplex.

Actually, it is more convenient to work with the short cyclic complex

$$(5.89)\qquad C^{\mathrm{ev}}(\mathcal{A})\overset{\partial}{\leftrightarrows}C^{\mathrm{odd}}(\mathcal{A}),\qquad \partial=b+B,$$

$$(5.90)\qquad C^{\mathrm{ev}}(\mathcal{A})=\bigoplus_{n\geq 0}C^{2n}(\mathcal{A}),\qquad C^{\mathrm{odd}}(\mathcal{A})\bigoplus_{n\geq 0}C^{2n+1}(\mathcal{A}),$$

where $\oplus$ is the algebraic direct sum, so that the cochains in the short complex have finite supports. We thus obtain two cohomology groups $HC^{\mathrm{ev}}(\mathcal{A})$ and $HC^{\mathrm{odd}}(\mathcal{A})$. For the reader familiar with [Co1] note that $HC^{\mathrm{ev}}(\mathcal{A})\oplus HC^{\mathrm{odd}}(\mathcal{A})$ is just the periodic cyclic cohomology seen as the filtration by dimension of the entire cyclic cohomology $HC^*_\epsilon(\mathcal{A})$.

We have a pairing between $HC^{\mathrm{ev}}(\mathcal{A})$ and $K_0(\mathcal{A})$ such that for a cyclic cocycle $\varphi=(\varphi_{2n})$ in $C^{\mathrm{odd}}(\mathcal{A})$ and for an idempotent $e\in\operatorname{Proj}M_k(\mathcal{A})$ we have

$$(5.91)\qquad \langle[\varphi],[e]\rangle=\sum_{|\geq|0}(-1)^n\frac{(2n)!}{n!}\varphi_{2n}\#\operatorname{tr}(e,\cdots,e),$$

where $\varphi_{2n}\#\operatorname{tr}$ is the$n+1$-linear map on $M_k(\mathcal{A})=M_k(\mathbb{C})\otimes\mathcal{A}$ defined by

$$\begin{aligned}&\varphi_{2n}\#\operatorname{tr}(\mu^0\otimes a^0,\cdots,\mu^{2n}\otimes a^{2n})\\ (5.92)\qquad &=\operatorname{tr}(\mu^0\ldots\mu^{2n})\varphi_{2n}(a^0,\cdots,a^{2n}),\qquad \mu^j\in M_k(\mathbb{C}),\quad a^j\in\mathcal{A}.\end{aligned}$$

(See [Co1, sect. IV.7.δ] using the table p. 371).

However we are dealing here with $\mathcal{A}=C^\infty(M)$ and this has important topological counterparts. First we can explicitly described the Hochschild cohomology in terms of de Rham's currents.

Proposition 5.23 ([Co2]) *1) We define an isomorphism $\psi\to C_\psi$ from the Hochschild cohomology group $H^k(\mathcal{A},\mathcal{A}^*)$ onto the space $\mathcal{D}'_k(M)$ of k-dimensional currents by letting*

(5.93)

$$\langle C_\psi,f^0df^1\ldots df^k\rangle=\frac{1}{k!}\sum_{\sigma\in S_k}\varepsilon(\sigma)\psi(f^0,f^1,\cdots,f^k),\qquad f^j\in\mathcal{A}.$$

2) Under this isomorphism we have $B=kd^t$ where d^t is the de Rham boundary for currents.

Note that if the k-cocycle ψ is completely antisymmetric in the last k variables then

$$\langle C_\psi, f^0 df^1 \dots df^k \rangle = \psi(f^0, f^1, \cdots, f^k), \qquad f^j \in \mathcal{A}. \tag{5.94}$$

The inverse of the above isomorphism is given by $C \to \psi_C$ where

$$\psi_C(f^0, f^1, \cdots, f^k) = \langle C_\psi, f^0 df^1 \dots df^k \rangle, \qquad f^j \in \mathcal{A}. \tag{5.95}$$

This defines a cocycle and we have

$$\begin{aligned} B_0\psi_C(f^0, \cdots, f^{k-1}) &= \psi_C(1, f^0, \cdots, f^{k-1}) \\ &= \langle C, df^0 \wedge \dots \wedge df^k \rangle = \langle d^t C, f^0 \wedge df^1 \wedge \dots \wedge df^k \rangle. \end{aligned} \tag{5.96}$$

Therefore given a current $C = (C_k) \in \oplus_{k \leq 0} \mathcal{D}'_k(M)$ we define a cochain in the short complex by letting

$$\varphi_C = (\varphi_{C_k}), \qquad \varphi_{C_k} = \frac{1}{k!} \psi_{C_k}. \tag{5.97}$$

Since $\partial \varphi_C = \varphi_{d^t C}$ this map induces a morphism $\omega \to \varphi_\omega$ from the even de Rham homology $H_{\text{ev}}(M) = \oplus_{n \leq 0} H_{2n}(M)$ of the manifold into the even cyclic cohomology $HC^{\text{ev}}(\mathcal{A})$ of $\mathcal{A}$.

Proposition 5.24 *1) The map $\omega \to \varphi_\omega$ is an isomorphism from $H_{ev}(M)$ onto $HC^{ev}(\mathcal{A})$.*

2) Let C be an even closed current on M and let $\mathcal{E}$ be an Hermitian vector bundle over M. Then

$$\langle [\varphi_C], [\mathcal{E}] \rangle = \langle [C], \operatorname{Ch}^* \mathcal{E} \rangle, \tag{5.98}$$

where the l.h.s. is the pairing of the cyclic cohomology class of φ_C with the class of $\mathcal{E}$ in $K_0(\mathcal{A}) \simeq K^0(M)$, while the r.h.s. is the pairing of the homology class of C with the Chern character of $\mathcal{E}$.

Proof. 1) Let $\varphi = (\varphi_{2n}) \in C^{\text{ev}}(\mathcal{A})$ such that $\partial\varphi = 0$, i.e.

$$B\varphi_{2n} + b\varphi_{2n} = 0. \tag{5.99}$$

For n large enough we have $\varphi_{2p} = 0$ for $p > n$. In particular $\varphi_{2n+2} = 0$ so that $b\varphi_{2n} = 0$. Thus φ_{2n} is a Hochschild cocycle and by proposition 5.23 there exists an unique $2n$ dimensional current $C_{2n} = (2n)! C_{\varphi_{2n}}$ on M such that $\varphi_{2n} - \varphi_{C_{2n}}$ is a cobounday cochain,

$$\varphi_{2n} = \varphi_{C_{2n}} + b\psi_{2n-1}, \qquad \psi_{2n-1} \in C^{2n-1}(\mathcal{A}). \tag{5.100}$$

This current is closed since the boundary d^t corresponds in Hochschild cohomology to the B operator and we have

$$B\varphi_{C_{2n}} = B_{\varphi_{2n}} + Bb\psi_{2n-1} = -b(\varphi_{2n-2} + B\psi_{2n-1}). \tag{5.101}$$

Then (5.96) implies $B\varphi_{C_{2n}} = 0$, so

$$b(\varphi_{2n-2} - B\psi_{2n-1}) = -B(\varphi_{2n} - b\psi_{2n-1}) = 0. \tag{5.102}$$

Hence $\varphi_{2n-2} - B\psi_{2n-1}$ is a Hochschild cocycle and there exists an unique $2n-2$ dimensional current C_{2n-2} and a cochain $\psi_{2n-3} \in C^{2n-3}(\mathcal{A})$ such that

$$\varphi_{2n-2} = \varphi_{C_{2n-2}} + B\psi_{2n-1} + b\psi_{2n-3}. \tag{5.103}$$

The current C_{2n-2} is also closed, since

$$B\varphi_{C_{2n-2}} = B\varphi_{2n-2} - Bb\psi_{2n-3} = b(\varphi_{2n-4} + B\psi_{2n-3}). \tag{5.104}$$

Moreover if ψ'_{2n-1} is another cochain satisfying (5.100) then $\psi_{2n-1} - \psi'_{2n-1}$ is a Hochschild cocycle and ψ'_{2n-1} yields another closed current $C'_{2n-2} \in \mathcal{D}'_{2n-2}(M)$ such that

$$\varphi_{C_{2n-2}} - \varphi_{C'_{2n-2}} = B(\psi_{2n-1} - \psi'_{2n-1}). \tag{5.105}$$

So $C_{2n-2} - C'_{2n-2}$ is a boundary current and the homology class of C_{2n-2} is uniquely determined.

Repeating this process we get an even current $C = (C_{2n})$ and an odd cochain $\psi = (\psi_{2n+1})$ such that

$$\varphi = \varphi_C + \partial\psi. \tag{5.106}$$

Each step yields an uniquely determined homology class of currents, so we have proven that for each class $\varphi \in HC^{\mathrm{ev}}(\mathcal{A})$ there exists an unique even homology class $\omega \in H_{\mathrm{ev}}(M)$ such that $\varphi = \varphi_\omega$. Thus $\omega \to \varphi_\omega$ is an isomorphism from $H_{\mathrm{ev}}(M)$ onto $HC^{\mathrm{ev}}(\mathcal{A})$.

2) Let $\mathcal{E}$ be an Hermitian vector bundle over M. As a $\mathcal{A}$-module $C^\infty(M,\mathcal{E})$ is isomorphic to $e\mathcal{A}^k$ for some integer k and some idempotent $e \in \operatorname{Proj} M_k(\mathcal{A})$. So we can suppose that $\mathcal{E} = \operatorname{im} e$ and define a connection on $\mathcal{E}$ by setting

$$\nabla = (e \otimes 1)d : C^\infty(M,\mathcal{E}) \longrightarrow C^\infty(M, \mathcal{E} \otimes \Lambda^1 T^*M). \tag{5.107}$$

One can check that the curvature of this connection is

$$\nabla^2 = e(de)^2 = e(de)^2 e \in C^\infty(M, \mathcal{E} \otimes \Lambda^2 T^*M), \tag{5.108}$$

so that the Chern character of $\mathcal{E}$ is represented by the even closed form

$$\operatorname{Tr} e^{-\nabla^2} = \sum \frac{(-1)^n}{n!} \operatorname{Tr}(e(de)^2)^n = \sum \frac{(-1)^n}{n!} \operatorname{Tr} e(de)^{2n}. \tag{5.109}$$

Now let $C = (C_{2n})$ be an even closed current. Then

$$\varphi_{C_{2n}} \# \operatorname{Tr}(a^0 \otimes \mu^0, \cdots, a^{2n} \otimes \mu^{2n}) = \frac{1}{(2n)!} \operatorname{Tr}(\mu^0 \cdots \mu^{2n}) \langle C_{2n}, a^0 da^1 \cdots a^{2n} \rangle$$
$$= \frac{1}{(2n)!} \langle C_{2n}, \operatorname{Tr}(a^0 \otimes \mu^0, \cdots, a^{2n} \otimes \mu^{2n}) \rangle. \tag{5.110}$$

Therefore

$$\langle [\varphi_C], [\mathcal{E}] \rangle = \sum \frac{(-1)^n}{n!} \langle C_{2n}, \operatorname{Tr} e(de)^{2n} \rangle = \langle [C], \operatorname{Ch}^* \mathcal{E} \rangle, \tag{5.111}$$

which concludes the proof. ■

Let us now go back to our index problem. The cyclic cohomology is actually the natural recipient, at least at the operatorial level, to a dual Chern character whose pairing enables us as in the Atiyah-Singer index theorem [AS] to compute the index of D with coefficient in $K_0(\mathcal{A})$.

Indeed let $\mathcal{H}$ be the Hilbert space

$$L^2(M, \mathcal{S}) = L^2(M, \mathcal{S}^+) \oplus L^2(M, \mathcal{S}^-). \tag{5.112}$$

Then $\mathcal{A} = C^\infty(M)$ acts on this Hilbert space by multiplication and D is an odd unbounded selfadjoint operator on $\mathcal{H}$. As D is an (Heisenberg) elliptic differential operator of order 1, it has compact resolvent and almost commutes with $\mathcal{A}$ to the extent that $[D, a]$ is bounded for any $a \in \mathcal{A}$. Hence $(\mathcal{A}, \mathcal{H}, D)$ is an even spectral triple in the sense of [CM2].

This spectral triple is $(d+2)$-summable since by theorem 4.11 we have

$$\mu_n(D^{-1}) = \mathrm{O}(n^{\frac{-1}{(d+2)}}). \tag{5.113}$$

Suppose now that the complex powers $|D|^z$, $z \in \mathbb{C}$, define a one parameter group of $\Psi_V DO$'s such that $\operatorname{ord}|D|^z = z$. By chapter 3 this is certainly true if D^2 is a sublaplacian. Then theorem 4.5 implies that if we let $\Sigma = \{k \in \mathbb{Z}; k \leq (d+2)\}$ then for any $\Psi_V DO$ with integral order ≤ 0 the function $\operatorname{TR} P|D|^{-z}$ is holomorphic on $\mathbb{C} \setminus \Sigma$ and has at most simple pole singularities on Σ with a residue at $z = 0$ given by

$$\operatorname{res}_{z=0} \operatorname{TR} P|D|^{-z} = \operatorname{Res} P. \tag{5.114}$$

It follows that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has a discrete and simple dimension spectrum contained in Σ. Thus it satisfies to the hypothesis of [CM2, theorem II.3] which provides us with a Chern character in cyclic cohomology given by local formulae.

Proposition 5.25 ([CM2]) *Suppose that D^2 is a sublaplacian. Then:*

1) We define an even cyclic cocycle $\varphi = (\varphi_{2n})$ as follows. For $n \neq 0$ the cochain φ_{2n} is given by

(5.115)
$$\varphi_{2n} = \sum_{\alpha} c_\alpha \operatorname{Res} \gamma a^0 [D, a^1]^{(\alpha_1)} \dots [D, a^{2n}]^{(\alpha_{2n})} |D|^{-2(|\alpha|+n)}, \qquad a^j \in \mathcal{A},$$

where $c_\alpha^{-1} = (-1)^{|\alpha|} 2\alpha!(\alpha_1 + 1) \cdots (\alpha_1 + \cdots + \alpha_{2n} + 2n)$ and the symbol $T^{(k)}$ denotes the k'th iterated commutator with D^2, while for $n = 0$ we have

$$\varphi_0(f) = \int_M f(x) \operatorname{Str}_{\mathcal{S}} a_0(D^2)(x), \qquad f \in \mathcal{A}, \tag{5.116}$$

where $a_0(D^2)(x)$ is the density occurring as the constant term on the asymptotics of the heat kernel on the diagonal.

2) The pairing with the class of φ in $HC^{ev}(\mathcal{A})$ gives the index with coefficients in $K_0(\mathcal{A})$,

Remark 5.26 The expression of φ_0 differs a little bit from [CM2] but it is the good one since it gives back the index of D and if D is not invertible $\operatorname{res}_{z=0} z \operatorname{TR} \gamma |D|^{-z}$ is not equal to the constant term in the asymptotics of $\operatorname{Trace} \gamma e^{-\epsilon D^2}$ for ϵ small.

Putting all these things together we obtain:

Theorem 5.27 *If D^2 is a sublaplacian, then:*

1) There exists an even homology class $\operatorname{Ch}_* D \in H_{ev}(M)$ such that for any Hermitian vector bundle over M with a Hermitian connection ∇ we have

$$\operatorname{ind} D_{\nabla,\mathcal{E}} = \langle \operatorname{Ch}_* D, \operatorname{Ch}^* \mathcal{E} \rangle. \tag{5.117}$$

2) We can define an explicit closed even current $C = (C_{2n})$ representing $\operatorname{Ch}_* D$ as follows. For $n \neq 0$ define C_{2n} by

$$\begin{aligned} &\langle C_{2n}, f^0 df^1 \wedge \ldots \wedge df^{2n} \rangle \\ &= (2n)! \sum_{\alpha} c_\alpha \operatorname{Res} \gamma f^0 [D, f^1]^{\alpha_1} \ldots [D, f^{2n}]^{\alpha_{2n}} |D|^{-2(|\alpha|+n)}, \end{aligned} \tag{5.118}$$

where $c_\alpha^{-1} = (-1)^{|\alpha|} 2\alpha!(\alpha_1 + 1) \cdots (\alpha_1 + \cdots + \alpha_{2n} + 2n)$ and the symbol $T^{(k)}$ denotes the k'th iterated commutator with D^2, while for $n = 0$ we have

$$\langle C_0, f \rangle = \int_M f(x) \operatorname{Str}_{\mathcal{S}} a_0(D^2)(x), \tag{5.119}$$

where $a_0(D^2)(x)$ is the density occurring as the constant term on the asymptotic of the heat kernel on the diagonal.

Remark 5.28 The only thing we need for applying the Connes-Moscovici theorem is that the complex powers $|D|^z$, $z \in \mathbb{C}$, define a one parameter group of $\Psi_{\mathcal{V}}DO$'s such that $\mathrm{ord}|D|^z = z$. It is also possible to prove this is true for any elliptic $\Psi_{\mathcal{V}}DO$ of order 1. So the above theorem holds for any odd selfadjoint elliptic $\Psi_{\mathcal{V}}DO$ of order 1.

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