ANALOGUES OF THE HOLOMORPHIC MORSE INEQUALITIES IN CR GEOMETRY

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This talk is a preliminary report about a joint project with George Marinescu on extending to the CR setting Demailly's holomorphic Morse inequalities together with some applications to complex geometry, including a generalization of the Grauert-Riemenschneider criterion to the noncompact setting.

The talk is divided into 3 sections. In Section 1 we briefly review the holomorphic Morse inequalities. In Section 2 we recall the main definitions and properties concerning CR manifolds, CR vector bundles, CR connections and the $\overline{\partial}_b$-complex. In Section 3 we present our main results.

1. Holomorphic Morse Inequalities

By Kodaira's embedding theorem a compact complex manifold is projective algebraic iff it carries a positive holomorphic line bundle. The Grauert-Riemenschneider conjecture was an attempt to generalize Kodaira's embedding theorem to compact Moishezon manifolds. Recall that the latter are compact complex manifolds which are projective algebraic up to a proper modification or, equivalently, have maximal Kodaira dimension.

Conjecture (Grauert-Riemenschneider)**.** *A compact complex manifold is Moishezon if it carries a holomorphic line bundle which is positive on a dense open set.*

This was conjecture was first proved by Siu ([Si1], [Si2]) using elliptic estimates together with the Hirzbruch-Riemann-Roch formula. Subsequently, Demailly [De] gave an alternative proof based on a holomorphic version of the classical Morse inequalities as follows.

Let M^n be a complex manifold and let L be a Hermitian holomorphic line bundle over M with curvature F^L. It is convenient to identify F^L with the section of $\operatorname{End} T_{0,1}$ such that $\frac{\partial}{\partial z^j} \rightarrow F(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k})\frac{\partial}{\partial z^k}$.

For $q = 0, \ldots, n$ we let O_q denote the open set consisting of points $x \in M$ such that $F^L(x)$ has q negative eigenvalues and $n-q$ positive eigenvalues and we set $O_{\leq q} = O_0 \cup \ldots \cup O_q$.

Theorem 1.1 (Demailly)**.** *As $k \rightarrow \infty$ the following asymptotics hold.*

(i) Weak Holomorphic Morse Inequalities:

$$\dim H^{0,q}(M, L^k) \leq (-1)^q (\frac{k}{2\pi})^n \int_{O_q} \det F^L + \mathrm{o}(k^n). \tag{1.1}$$

(ii) Strong Holomorphic Morse Inequalities:

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^{0,j}(M, L^k) \leq (-1)^q (\frac{k}{2\pi})^n \int_{O_{\leq q}} \det F^L + \mathrm{o}(k^n). \tag{1.2}$$

(iii) Asymptotic Hirzbruch-Riemann-Roch formula:

$$\chi(M,L^k)=\sum_{j=0}^{n}(-1)^j\dim H^{0,j}(M,L^k)=(\frac{k}{2\pi})^n\int_M \det F^L+\mathrm{o}(k^n), \tag{1.3}$$

where $\chi(M,L^k)$ is the holomorphic Euler characteristic with coefficients in L^k.

In particular, for $q=1$ we get

$$-\dim H^{0,0}(M,L^k)+H^{0,1}(M,L^k)\leq\frac{-1}{n!}(\frac{k}{2\pi})^n\int_{O_{\leq 1}}\det F^L+\mathrm{o}(k^n). \tag{1.4}$$

Thus,

$$\dim H^{0,0}(M,L^k)\geq\frac{1}{n!}(\frac{k}{2\pi})^n\int_{O_{\leq 1}}\det F^L+\mathrm{o}(k^n). \tag{1.5}$$

If $\int_{O_{\leq 1}}\det F^L>0$ (e.g. if L is semi-positive and is >0 at a point) then we get:

$$\dim H^{0,0}(M,L^k)\gtrsim k^n, \tag{1.6}$$

which implies that M has maximal Kodaira dimension, i.e., M is Moishezon.

In [Bi] Bismut gave a heat kernel proof of Demailly's inequalities. Bismut's approach can be divided into 2 main steps.

Step 1: For $q=0,\ldots,n$ let $\Delta^{0,q}_{L^k}$ denote the Dolbeault Laplacian acting on sections of $\Lambda^{0,q}T^*M\otimes L^k$. We let $\mathcal{F}^L$ be the Clifford lift of F^L, i.e, the section of $\mathrm{End}(\Lambda^{0,*}T^*M)$ so that locally we have $\mathcal{F}^L=F(\frac{\partial}{\partial z^j},\frac{\partial}{\partial z^k})\varepsilon(d\bar{z^j})\iota(dz^k)$. Then Bismut proved:

Theorem 1.2 (Bismut)**.** *For any $t>0$ we have*

$$\mathrm{Tr}\, e^{-\frac{t}{k}\Delta^{0,q}_{L^k}}=(\frac{k}{2\pi})^n\int_M\det[\frac{F^L}{1-e^{-tF^L}}]\,\mathrm{Tr}_{|\Lambda^{0,q}}\,e^{-t\mathcal{F}^L}+\mathrm{o}(k^n). \tag{1.7}$$

Step 2: By taking the limit as $t\rightarrow\infty$ in the integral in (1.7) Bismut recovered the inequalities (1.1)–(1.3), via linear-algebraic arguments similar to that of his earlier proof of the Morse inequalities.

2. CR manifolds and the $\overline{\partial}_b$-complex

2.1. **CR Manifolds.** A CR structure on an orientable manifold M^{2n+1} is given by a rank n vector bundle $T_{1,0}\subset T_{\mathbb{C}}M$ such that:

(i) $T_{1,0}$ is integrable in Froebenius' sense;

(ii) $T_{1,0}\cap T_{0,1}=\{0\}$, where $T_{0,1}=\overline{T_{1,0}}$.

The main examples of CR manifolds include:

- Boundaries of complex domains;
- Circle bundles over complex manifolds;
- Boundaries of complex hyperbolic spaces.

Given be a global non-vanishing real 1-form θ annihilating $T_{1,0}\oplus T_{0,1}$ the associated Levi form is given by

$$L_\theta(Z,W)=-id\theta(Z,\overline{W}),\quad Z,W\in C^\infty(M,T_{1,0}). \tag{2.1}$$

We say that M is *strictly pseudoconvex* when we can choose θ so that at every point L_θ is positive definite. Similarly, we say M is *κ-strictly pseudoconvex* when we can choose θ so that at every point L_θ has exactly κ-negative eigenvalues and $n-\kappa$ positive eigenvalues.

2.2. **The $\overline{\partial}_b$-complex.** Let $\mathcal{N}$ be a supplement of $T_{1,0}\oplus T_{0,1}$ in $T_{\mathbb{C}}M$ and define:

$$\Lambda^{1,0} = \text{annihilator in } T^*_{\mathbb{C}}M \text{ of } T_{0,1}\oplus\mathcal{N},$$
$$\Lambda^{0,1} = \text{annihilator in } T^*_{\mathbb{C}}M \text{ of } T_{1,0}\oplus\mathcal{N},$$
$$\Lambda^{p,q} = (\Lambda^{1,0})^p\wedge(\Lambda^{0,1})^q, \quad p,q=0,\ldots,n.$$

This gives rise to the splitting,

$$\Lambda^*T^*_{\mathbb{C}}M = \Big(\bigoplus_{p,q=0}^{n}\Lambda^{p,q}\Big)\oplus(\theta\wedge\Lambda^*T^*_{\mathbb{C}}M). \tag{2.2}$$

If $\alpha\in C^\infty(M,\Lambda^{0,q})$, then we can write

$$d\alpha = \partial_b\alpha+\overline{\partial}_b\alpha+\theta\wedge\beta, \tag{2.3}$$

with $\partial_b\alpha\in C^\infty(M,\Lambda^{1,q})$ and $\overline{\partial}_b\alpha\in C^\infty(M,\Lambda^{0,q+1})$.

We have $\overline{\partial}_b^2=0$, so $\overline{\partial}_b : C^\infty(M,\Lambda^{0,*})\rightarrow C^\infty(M,\Lambda^{0,*+1})$ is a chain complex whose cohomology groups are denoted $H_b^{0,q}(M)$, $q=0,\ldots,n$.

Endowing $T_{\mathbb{C}}M$ with a Hermitian metric, the Kohn Laplacian is

$$\Box_b = \overline{\partial}_b^*\overline{\partial}_b+\overline{\partial}_b\overline{\partial}_b^*. \tag{2.4}$$

Proposition 2.1. *We have* $H_b^{0,q}(M)\simeq\ker\Box_{b,q}$.

For $x\in M$ let $\kappa_+(x)$and $\kappa_-(x)$ be the number of positive and negative eigenvalues of the Levi form L_θ at x.

Definition 2.2 (Condition $Y(q)$)**.** *The condition $Y(q)$ is satisfied when for all $x\in M$ we have:*

$$q\notin\{\kappa_-(x),\ldots,n-\kappa_+(x)\}\cup\{\kappa_+(x),\ldots,n-\kappa_-(x)\}. \tag{2.5}$$

Examples. 1) If M is strictly pseudoconvex then the condition $Y(q)$ means $q\neq 0,n$.

2) If M is κ-strictly pseudoconvex then the condition $Y(q)$ means $q\neq\kappa,n-\kappa$.

3) The condition $Y(0)$ means that L_θ has at least one positive and one negative eigenvalue.

Proposition 2.3 (Kohn)**.** *Under condition $Y(q)$ the operator $\Box_{b,q}$ is hypoelliptic with gain of 1 derivative, i.e., for any compact $K\subset M$ we have estimates,*

$$\|u\|_{s+1}\leq C_{Ks}\|\Box_{b,q}u\|_s \quad \forall u\in C_K^\infty(M,\Lambda^{0,q}). \tag{2.6}$$

Corollary 2.4. *If the condition $Y(q)$ holds then* $\dim H_b^{0,q}(M)<\infty$.

2.3. **CR vector bundles and CR connections.** In the sequel we say that a map $\phi=(\phi_{kl}):M\rightarrow M_p(\mathbb{C})$ is CR when $\overline{\partial}_b\phi_{kl}=0$.

Definition 2.5. *A CR vector bundle $\mathcal{E}$ over M is a vector bundle given by a covering of M by trivializations $\tau_i:\mathcal{E}_{|U_j}\rightarrow U_j\times\mathbb{C}^p$ whose transition maps $\tau_{ij}=\tau_i\circ\tau_j^{-1}:U_i\cap U_j\rightarrow \mathrm{GL}_p(\mathbb{C})$ are CR maps.*

Given a vector bundle $\mathcal{E}$ over M for $p,q=0,\ldots,n$ we let $\Lambda^{p,q}(\mathcal{E})=\Lambda^{p,q}\otimes\mathcal{E}$.

Proposition 2.6. *If $\mathcal{E}$ is a CR vector bundle then there exists a unique operator,*

$$\overline{\partial}_{b,\mathcal{E}} : C^\infty(M, \Lambda^{0,*}(\mathcal{E})) \to C^\infty(M, \Lambda^{0,*+1}(\mathcal{E})), \tag{2.7}$$

such that $\overline{\partial}_{b,\mathcal{E}}^2 = 0$ and for any local CR frame $e_1, \ldots, e_p$ of $\mathcal{E}$ and any section $s = \sum s_i e_i$ we have

$$\overline{\partial}_{b,\mathcal{E}} s = \sum (\overline{\partial}_b s_i) \otimes e_i. \tag{2.8}$$

The cohomology groups of the complex $\overline{\partial}_{b,\mathcal{E}} : C^\infty(M, \Lambda^{0,*}(\mathcal{E})) \to C^\infty(M, \Lambda^{0,*+1}(\mathcal{E}))$ are denoted $H_b^{0,q}(M, \mathcal{E})$, $q = 0, \ldots, n$. As before if the condition $Y(q)$ holds then $\dim H_b^{0,q}(M, \mathcal{E}) < \infty$.

Next, let $\mathcal{E}$ be a CR vector bundle endowed with a Hermitian metric and let $\nabla : C^\infty(M, \mathcal{E}) \to C^\infty(M, T^*M \otimes \mathcal{E})$ be a connection. Recall that ∇ is said to be unitary when we have

$$d\langle \xi, \eta \rangle = \langle \nabla \xi, \eta \rangle + \langle \xi, \nabla \eta \rangle \tag{2.9}$$

for sections ξ and η of $\mathcal{E}$.

On the other hand, thanks to the splitting we can write:

$$\nabla = \nabla^{1,0} + \nabla^{0,1} + \theta \wedge D, \tag{2.10}$$

where $= \nabla^{1,0}$ and $\nabla^{0,1}$ map to sections of $\Lambda^{1,0}(\mathcal{E})$ and $\Lambda^{0,1}(\mathcal{E})$ respectively.

Definition 2.7. *∇ is a CR connection when $\nabla^{0,1} = \overline{\partial}_{b,\mathcal{E}}$.*

Now, let $\operatorname{End}_{\mathrm{sa}} \mathcal{E}$ the bundle of selfadjoint endomorphisms of $\mathcal{E}$. Then we have:

Proposition 2.8. *The space of unitary CR connections is a non-empty affine space modelled on $i\theta \otimes C^\infty(M, \operatorname{End}_{sa} \mathcal{E})$.*

3. CR Morse Inequalities

Let M^{2n+1} be a compact CR manifold together with a Hermitian metric on $T_{\mathbb{C}}M$ (not necessarily a Levi metric) and with a global real non-vanishing 1-form θ annihilating $T_{1,0} \oplus T_{0,1}$ and let L is a Hermitian CR line bundle over M with unitary CR connection of curvature F^L.

Our goal is to obtain analogues of the asymptotics (1.1)–(1.7) in this setting. There are several earlier related results in this direction.

First, in [Ge] Getzler proved an analogue of heat kernel asymptotics (1.7) for strictly pseudoconvex CR manifolds with Levi metric and conjectured that such an asymptotics should hold for more general CR manifolds. Nevertheless, he didn't derive asymptotic inequalities for $\dim H_b^{0,q}(M, L^k)$. There seems to be a mistake in Getzler's final formula (compare Theorem 3.1 below).

Later on, as a consequence of his version of the holomorphic Morse inequalities for pseudoconcave complex manifolds, Marinescu [Ma] obtained a lower bound for $H_b^{0,0}(M, L^{\otimes k})$ when M is the boundary of a strictly q-concave domain on a q-concave complex manifold X^{2n} with $n \geq 3$ and $q \leq n-2$.

In addition, Berman [Be] proved a version of Demailly's inequalities for complex manifold with nondegenerate boundary and Fu has announced during his talk at the symposium analogues of the weak holomorphic Morse inequalities (1.1) on bounded finite type pseudoconvex domains in $\mathbb{C}^2$.

3.1. **Heat kernel version.** Let $\Box^{0,q}_{b,L^k}$ be the Kohn Laplacian acting on sections of $\Lambda^{0,q}(L^k)$. As before it will be convenient to identify F^L and L_θ with the sections of $\operatorname{End}_{\mathbb{C}} T_{1,0}$ such that, for any orthonormal frame $Z_1, \ldots, Z_n$ of $T_{1,0}$, we have:

$$F^L Z_j = F^L(Z_j, Z_{\bar{k}})Z_k, \quad L_\theta Z_j = L_\theta(Z_j, Z_k)Z_k. \tag{3.1}$$

Furthermore, for $\mu \in \mathbb{R}$ we set

$$F^L_\theta(\mu) = F^L - \mu L_\theta, \tag{3.2}$$

and we let $\mathcal{F}^L_\theta(\mu)$ denote the Clifford lift of $F^L_\theta(\mu)$ to $\Lambda^{0,*}$, i.e., the section of $\operatorname{End}_{\mathbb{C}} \Lambda^{0,*}$ such that, for any orthonormal frame $Z_1, \ldots, Z_n$ of $T_{1,0}$ with dual coframe $\theta^1, \ldots, \theta^n$, we have

$$\mathcal{F}^L_\theta(\mu) = [F^L(Z_j, \overline{Z_k}) - \mu L_\theta(Z_j, Z_k)]\varepsilon(\theta^{\bar{j}})\iota(\theta^k). \tag{3.3}$$

Theorem 3.1 (GM+RP). *Assume that the condition $Y(q)$ holds. Then for any $t > 0$ we have*

$$\operatorname{Tr} e^{-t\Box^{0,q}_{b,L^k}} = (\frac{k}{4\pi})^{n+1} \int_M G^{0,q}(x,t)d\nu(x) + \mathrm{O}(k^n), \tag{3.4}$$

$$G^{0,q}(x,t) = \int_{-\infty}^{\infty} \det[\frac{F^L_\theta(\mu)}{1 - e^{-tF^L_\theta(\mu)}}] \operatorname{Tr} e^{-t\mathcal{F}^L_\theta(\mu)} d\mu, \tag{3.5}$$

where $d\nu(x)$ denotes the volume form of M.

Remark 3.2. We actually have a complete and local asymptotics in k, so this might yield a CR analogue of the Tian-Yau-Zelditch-Catlin asymptotics on $(0,q)$-forms.

3.2. **Cohomological version (in progress).** We make the following extra assumptions:

- M is κ-strictly pseudoconvex;
- We can choose F^L and $d\theta$ and the Hermitian metric of $T_{\mathbb{C}}M$ so that we have

$$[F^L, L_\theta] = 0. \tag{3.6}$$

This condition is automatically satisfied when M is strictly pseudoconvex by taking the metric to be the Levi metric.

Proposition 3.3 (GM+RP). *Under the above assumptions for $q \neq \kappa, n-\kappa$ we have:*

$$\lim_{t\to\infty} G^{0,q}(x,t) = (-1)^q \int_{\lambda_q(x)}^{\lambda_{q+1}(x)} \det(L_\theta^{-1}F^L(x) - \mu)d\mu, \tag{3.7}$$

where $\lambda_j(x)$ denotes the j'th eigenvalue of $L_\theta^{-1}F^L(x)$ counted with multiplicity.

Thanks to this result we may argue as in [Bi] to get:

Proposition 3.4 (GM+RP). *Under the same assumptions for $q \neq \kappa, n-\kappa$ we have:*

1) If $\lambda_{q+1}(x_0) > \lambda_q(x_0)$ for some $x_0 \in M$, then we have:

$$\dim H^{0,q}(M, L^k) \gtrsim k^{n+1}. \tag{3.8}$$

2) If $\lambda_{q+1}(x) = \lambda_q(x)$ at every point, then we have:

$$\dim H^{0,q}(M, L^k) = 0(k^n). \tag{3.9}$$

3.3. **Application to complex geometry (in progress).** As an application to the previous results we obtain:

Theorem 3.5. *Let M be a complex manifold (not necessarily compact) together with a Hermitian holomorphic line bundle L such that:*

(i) L is positive outside a Stein domain;

(ii) F^L degenerates with multiplicity at least 2 on ∂D.

Then M is Moishezon.

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