Introduction to Noncommutative Geometry
Part 4: Index Theory and Connes-Chern Character

Raphaël Ponge

Seoul National University & UC Berkeley
www.math.snu.ac.kr/∼ponge

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Overview of Noncommutative Geometry

Classical	NCG
Riemannian Manifold (M, g)	Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$
Vector Bundle E over M	Projective Module $\mathcal{E}$ over $\mathcal{A}$ $\mathcal{E} = e\mathcal{A}^q, \;\; e \in M_q(\mathcal{A}), \; e^2 = e$
$\mathrm{ind} \not{D}_{\nabla^E}$	$\mathrm{ind}\ D_{\nabla^{\mathcal{E}}}$
de Rham Homology/Cohomology	Cyclic Cohomology/Homology
Atiyah-Singer Index Formula $\mathrm{ind} \not{D}_{\nabla^E} = \int \hat{A}(R^M) \wedge \mathrm{Ch}(F^E)$	Connes-Chern Character $\mathrm{Ch}(D)$ $\mathrm{ind}\ D_{\nabla^{\mathcal{E}}} = \langle \mathrm{Ch}(D), \mathrm{Ch}(\mathcal{E}) \rangle$
Local Index Theorem	CM cocycle

Cyclic Homology

Hochschild Homology $\mathrm{HH}_\bullet(\mathcal{A})$	$(C_\bullet(\mathcal{A}), b)$
Cyclic Homology $H^\lambda_\bullet(\mathcal{A}) \simeq \mathrm{HC}_\bullet(\mathcal{A})$	$\big(C^\lambda_\bullet(\mathcal{A}), b\big)$ and $\big(C_\bullet(\mathcal{A}), b, B\big)$
Periodic Cyclic Homology $\mathrm{HP}_\bullet(\mathcal{A})$	$\big(C_{[\bullet]}(\mathcal{A}), b + B\big)$
Chern Character	$\mathrm{Ch}(\mathcal{E}) \in \mathrm{HP}_0(\mathcal{A})$

$$C_m(\mathcal{A}) = \mathcal{A}^{\otimes(m+1)},$$

$$C_{\bullet-1}(\mathcal{A}) \xleftarrow{\ b\ } C_\bullet(\mathcal{A}) \xrightarrow{\ B\ } C_{\bullet+1}(\mathcal{A}), \quad b^2 = B^2 = bB + Bb = 0,$$

$$C_{[i]}(\mathcal{A}) = \prod_{q\geq 0} C_{2q+i}(\mathcal{A}), \quad i = 0, 1.$$

Cyclic Cohomology

Cyclic Cohomology

Hochschild Cohomology $\mathrm{HH}^\bullet(\mathcal{A})$	$(C^\bullet(\mathcal{A}), b)$
Cyclic Cohomology $H^\bullet_\lambda(\mathcal{A}) \simeq \mathrm{HC}^\bullet(\mathcal{A})$	$(C^\bullet_\lambda(\mathcal{A}), b)$ and $(C^\bullet(\mathcal{A}), b, B)$
Periodic Cyclic Cohomology $\mathrm{HP}^\bullet(\mathcal{A})$	$\big(C^{[\bullet]}(\mathcal{A}), b + B\big)$
Connes-Chern Character	$\mathrm{Ch}(D) \in \mathrm{HP}^0(\mathcal{A})$

$$C^m(\mathcal{A}) = \{(m+1)\text{-linear forms } \varphi : \mathcal{A}^{m+1} \to \mathbb{C}\} = \left(\mathcal{A}^{\otimes(m+1)}\right)^*,$$

$$C^m_\lambda(\mathcal{A}) = \{\varphi \in C^m(\mathcal{A});\ \varphi \text{ cyclic}\},$$

$$C^{\bullet-1}(\mathcal{A}) \xleftarrow{B} C^\bullet(\mathcal{A}) \xrightarrow{b} C^{\bullet+1}(\mathcal{A}), \quad b^2 = B^2 = bB + Bb = 0.$$

Cyclic Cochains and Connes' B-Operator

Definition

1. The cyclic operator $T : C^m(\mathcal{A}) \to C^m(\mathcal{A})$ is given by
$$(T\varphi)(a^0, \ldots, a^m) = (-1)^m \varphi(a^m, a^0, \ldots, a^m), \;\; \varphi \in C^m(\mathcal{A}).$$
2. A cochain $\varphi \in C^m(\mathcal{A})$ is cyclic when $T\varphi = \varphi$.
3. The operator $B : C^\bullet(\mathcal{A}) \to C^{\bullet-1}(\mathcal{A})$ is the composition,
$$B = AB(1 - T), \quad A = 1 + T + \cdots + T^{m-1},$$
$$(B_0\varphi)(a^0, \ldots, a^m) = \varphi(1, a^0, \ldots, a^m), \quad \varphi \in C^m(\mathcal{A}).$$

Lemma (Connes)

1. *B is annihilated by the cyclic cochains.*
2. $B^2 = 0$ *and* $Bb + bB = 0$.

Definition

The periodic cyclic cohomology is the cohomology of the complex,

$$C^{[0]}(\mathcal{A}) \overset{b+B}{\rightleftarrows} C^{[1]}(\mathcal{A}), \quad \text{where } C^{[i]}(\mathcal{A}) = \bigoplus_{q\geq 0} C^{2q+i}(\mathcal{A}).$$

Remarks

1. An even periodic cocycle is a finite sequence $\varphi = (\varphi_{2q})_{q\geq 0}$, $\varphi_{2q} \in C^{2q}(\mathcal{A})$, such that

$$\varphi_{2q} = 0 \text{ for } q \gg 1 \quad \text{and} \quad b\varphi_{2q} + B\varphi_{2q+2} = 0 \ \forall q \geq 0.$$

2. An even cyclic cocycle $\tau \in C^{2q}_{\lambda}(\mathcal{A})$, $b\tau = 0$, is naturally identified with the even periodic cocycle,

$$(0, \ldots, 0, \tau, 0, \ldots) \in C^{[0]}(\mathcal{A}), \quad b\tau = 0, \ B\tau = 0.$$

3. There are similar remarks for odd cochains.

Example: $\mathcal{A} = \mathcal{A}_\theta$

Noncommutative Torus

- Noncommutative torus:

$$\mathcal{A}_\theta = \mathcal{A}_\theta = \left\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n ; (a_{m,n}) \in \mathcal{S}(\mathbb{Z}^2) \right\},$$

$$U^* = U^{-1}, \quad V^* = V^{-1}, \quad VU = e^{-2i\pi\theta} UV.$$

- Unique normalized trace:

$$\tau_0\left(\sum a_{m,m} U^m V^n \right) = a_{00}.$$

- Basic derivations:

$$\delta_1(U^m V^n) = mU^m V^n \quad \text{and} \quad \delta_2(U^m V^n) = nU^m V^n.$$

Example: $\mathcal{A} = \mathcal{A}_\theta$

Theorem (Connes)

1. *We have* $\dim \mathrm{HP}^0(\mathcal{A}_\theta) = \dim \mathrm{HP}^1(\mathcal{A}_\theta) = 2$.
2. *A basis of* $\mathrm{HP}^1(\mathcal{A}_\theta)$ *is given by the cyclic 1-cocycles,*

$$\varphi_j(a^0, a^1) = \tau_0\left(a^0 \delta_j(a^1)\right), \quad j = 1, 2.$$

3. *A basis of* $\mathrm{HP}^0(\mathcal{A}_\theta)$ *is given by the canonical trace* τ_0 *and the cyclic 2-cocycle,*

$$\tau_2(a^0, a^1, a^2) = \tau_0\left[a^0\left(\delta_1(a^1)\delta_2(a^2) - \delta_2(a^1)\delta_1(a^2)\right)\right].$$

Twisted Dirac Operators

Setup

- M^n is a compact spin (oriented) Riemannian manifold (n even).
- E is a Hermitian vector bundle with Hermitian connection ∇^E.
- $\not{D} : C^\infty(M, \$) \to C^\infty(M, \$)$ is the Dirac operator on M.

Definition

The operator $\not{D}_{\nabla^E} : C^\infty(M, \$ \otimes E) \to C^\infty(M, \$ \otimes E)$ is defined by

$$\not{D}_{\nabla^E} = \not{D} \otimes 1_E + (c \otimes 1_E)(1_\$ \otimes \nabla^E),$$

where $(c \otimes 1_E)(1_\$ \otimes \nabla^E)$ is given by the composition

$$C^\infty(M, \$ \otimes E) \xrightarrow{1_\$ \otimes \nabla^E} C^\infty(M, \$ \otimes T^*M \otimes E) \xrightarrow{c \otimes 1_E} C^\infty(M, \$ \otimes E),$$

and $c(\sigma \otimes \omega) = c(\omega)\sigma$ (Clifford action).

Fredholm Indices of Dirac Operators

Proposition

1. *The operator $\not{D}_{\nabla^E}$ is selfadjoint and takes the form,*

$$\not{D}_{\nabla^E} = \begin{pmatrix} 0 & \not{D}^-_{\nabla^E} \\ \not{D}^+_{\nabla^E} & 0 \end{pmatrix},$$

where $\not{D}^{\pm}_{\nabla^E} : C^\infty(M, \$^{\pm} \otimes E) \rightarrow C^\infty(M, \$^{\mp} \otimes E)$.

2. *The operator $\not{D}_{\nabla^E}$ is an elliptic differential operator, and hence is a Fredholm operator.*

Definition

The Fredholm index of $\not{D}_{\nabla^E}$ is defined by

$$\begin{aligned} \operatorname{ind} \not{D}_{\nabla^E} &:= \operatorname{ind} \not{D}^+_{\nabla^E} \\ &= \dim \ker \not{D}^+_{\nabla^E} - \dim \ker \not{D}^-_{\nabla^E}. \end{aligned}$$

Theorem (Atiyah-Singer '60s)

We have

$$\operatorname{ind} \not{D}_{\nabla^E} = \int_M \hat{A}(R^M) \wedge \mathrm{Ch}(F^E),$$

where

$$\hat{A}(R^M) = \det^{\frac{1}{2}}\left[\frac{R^M/2}{\sinh(R^M/2)}\right] \quad \textit{and} \quad \mathrm{Ch}(F^E) = \mathrm{Tr}\left[\exp(-F^E)\right].$$

Local Index Theorem

Mckean-Singer Formula

We have

$$\mathrm{ind}\, \not{D}_{\nabla^E} = \mathrm{Str}\left[e^{-t\not{D}^2_{\nabla^E}} \right] \qquad \forall t > 0,$$

where $\mathrm{Str} = \mathrm{Tr}_{|\not{S}^+\otimes E} - \mathrm{Tr}_{|\not{S}^-\otimes E}$.

Local Index Theorem (Patodi, Gilkey, Atiyah-Bott-Patodi)

We have

$$\lim_{t\to 0^+} \mathrm{Str}\left[e^{-t\not{D}^2_{\nabla^E}} \right] = \int_M \hat{A}(R^M) \wedge \mathrm{Ch}(F^E).$$

Combining this with the McKean-Singer formula proves the Atiyah-Singer index theorem.

Remark

We obtain a purely analytical proof of the LIT by using Getzler's rescaling.

Spectral Triples

Definition (Connes-Moscovici)

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of

1. A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
2. A $*$-algebra $\mathcal{A}$ represented in $\mathcal{H}$.
3. A selfadjoint unbounded operator D on $\mathcal{H}$ such that
    1. D maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
    2. $(D \pm i)^{-1}$ is compact.
    3. $[D, a]$ is bounded for all $a \in \mathcal{A}$.

Connections over a Spectral Triple

Setup

- $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple.
- $\mathcal{E}$ is finitely generated projective (right) module over $\mathcal{A}$.

Definition

The space of noncommutative 1-forms is

$$\Omega^1_D(\mathcal{A}) = \mathrm{Span}\{adb;\ a, b \in \mathcal{A}\} \subset \mathcal{L}(\mathcal{H}),$$

where $db = [D, b]$.

Definition

A connection on $\mathcal{E}$ is a linear map $\nabla^{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$ such that

$$\nabla^{\mathcal{E}}(\xi a) = \xi \otimes da + \left(\nabla^{\mathcal{E}} \xi\right) a \qquad \forall a \in \mathcal{A}\ \forall \xi \in \mathcal{E}.$$

Example: Dirac Spectral Triple

Setup

- $(C^\infty(M), L^2(M, \$), D\!\!\!/)$ is a Dirac spectral triple.
- E is a vector bundle over M and $\mathcal{E} = C^\infty(M, E)$.

Proposition

1. *We have*

$$\Omega^1_{D\!\!\!/}(C^\infty(M)) = c\,(C^\infty(M, T^*M)) = C^\infty(M, \operatorname{End}\$).$$

2. *If ∇^E is a connection on E, then the composition*

$$\nabla^{\mathcal{E}} = (c \otimes 1_E) \circ \nabla^E : C^\infty(M, E) \to C^\infty(M, (\operatorname{End}\$) \otimes E)$$

defines a connection on the f.g.. projective module $\mathcal{E} = C^\infty(M, E)$.

Example: Grassmannian Connection

Setup

- $(\mathcal{A}, \mathcal{H}, D)$ is an arbitrary spectral triple.
- $\mathcal{E} = e\mathcal{A}^q$, $e^2 = e \in M_q(\mathcal{A})$.

Definition

The Grassmannian connection $\nabla_0^{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A})$ is given by the composition,

$$\mathcal{E} \hookrightarrow \mathcal{A}^q \xrightarrow{d=[D,\cdot]} \Omega_D^1(\mathcal{A})^q \simeq \mathcal{A}^q \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A}) \xrightarrow{e\otimes 1} \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A}).$$

Proposition

1. *The Grassmannian connection is a connection on $\mathcal{E}$.*
2. *The set of connections on $\mathcal{E}$ is a nonempty affine space modelled on* $\mathrm{Hom}_{\mathcal{A}}\left(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A})\right)$.

Coupling with Connections

Setup

- $\mathcal{E}$ is a finitely generated projective module over $\mathcal{A}$.
- $\nabla^{\mathcal{E}}$ is a connection on $\mathcal{E}$.

Definition

The operator $D_{\nabla^{\mathcal{E}}} : \mathcal{E} \otimes_{\mathcal{A}} \operatorname{dom} D \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ is given by

$$D_{\nabla^{\mathcal{E}}}(\xi \otimes \zeta) = \xi \otimes D\zeta + c(\nabla^{\mathcal{E}})(\xi \otimes \zeta),$$

where $c(\nabla^{\mathcal{E}})$ is the composition,

$$\mathcal{E} \otimes \mathcal{H} \xrightarrow{\nabla^{\mathcal{E}} \otimes \mathrm{id}_{\mathcal{H}}} \mathcal{E} \otimes \Omega^1_D(\mathcal{A}) \otimes \mathcal{H} \xrightarrow{\mathrm{id}_{\mathcal{E}} \otimes c} \mathcal{E} \otimes \mathcal{H},$$

where $c(\omega \otimes \zeta) = \omega(\zeta)$ (recall that $\Omega^1_D(\mathcal{A}) \subset \mathcal{L}(\mathcal{H})$).

Fredholm Index

Remark

The operator $D_{\nabla^{\mathcal{E}}}$ takes the form,

$$D_{\nabla^{\mathcal{E}}} = \begin{pmatrix} 0 & D^-_{\nabla^{\mathcal{E}}} \\ D^+_{\nabla^{\mathcal{E}}} & 0 \end{pmatrix}, \quad D^{\pm}_{\nabla^{\mathcal{E}}} : \mathcal{E} \otimes \operatorname{dom} D^{\pm} \to \mathcal{E} \otimes \mathcal{H}^{\mp}.$$

Proposition

1. *The operator $D_{\nabla^{\mathcal{E}}}$ is Fredholm.*
2. *The Fredholm indices $\operatorname{ind} D^{\pm}_{\nabla^{\mathcal{E}}}$ do not depend on the choice of the connection $\nabla^{\mathcal{E}}$ and agree up to sign factor.*

Definition

The Fredholm index of D_{∇^E} is defined by

$$\begin{aligned} \operatorname{ind} D_{\nabla^{\mathcal{E}}} &:= \operatorname{ind} D^+_{\nabla^{\mathcal{E}}} \\ &= \dim\ker D^+_{\nabla^{\mathcal{E}}} - \dim\ker \left(D^+_{\nabla^{\mathcal{E}}}\right)^* . \end{aligned}$$

Assumption

- $(\mathcal{A}, \mathcal{H}, D)$ is p^+-summable, i.e., $\mu_n(D^{-1}) = \mathrm{O}(n^{-\frac{1}{p}})$, $p \geq 1$.

Lemma

Assume $\mathcal{E} = e\mathcal{A}^q$, $e^2 = e \in M_q(\mathcal{A})$, and $\nabla^{\mathcal{E}} = \nabla_0^{\mathcal{E}}$ is the Grassmanian connection. Then

$$\operatorname{ind} D_{\nabla_0^{\mathcal{E}}} = \frac{1}{2} \operatorname{Str} \left\{ (D^{-1}[D, e])^{2k+1} \right\} \quad \forall k \geq \frac{1}{2} p,$$

where $\operatorname{Str} = \operatorname{Tr}_{|\mathcal{H}^+} - \operatorname{Tr}_{|\mathcal{H}^-}$.

Connes-Chern Character

Observation

For $k \geq \frac{1}{2}p$, let $\tau_{2k}^{D} \in C^{2k}(\mathcal{A})$ be the cochain defined by

$$\tau_{2k}^{D}(a^0, \ldots, a^{2k}) = c_k \,\mathrm{Str}\left\{ D^{-1}[D, a^0] \cdots D^{-1}[D, a^{2k}] \right\}, \quad a^j \in \mathcal{A},$$

where $c_k = \frac{1}{2}(-1)^k \frac{k!}{(2k)!}$. Let $e \in \mathcal{A}$, $e^2 = e$. Then

$$\begin{aligned}
\langle \tau_{2k}^{D}, \mathrm{Ch}(e) \rangle &= \langle \tau_{2k}^{D}, \mathrm{Ch}_{2k}(e) \rangle, \\
&= (-1)^k \frac{(2k)!}{k!} \langle \tau_{2k}^{D}, (e - \frac{1}{2}) \otimes e^{\otimes (2k)} \rangle, \\
&= \frac{1}{2} c_k^{-1} \left\{ \tau_{2k}^{D}(e, e, \ldots, e) - \frac{1}{2} \tau_{2k}^{D}(1, e, \ldots, e) \right\}, \\
&= \frac{1}{2} \mathrm{Str}\left\{ (D^{-1}[D, e])^{2k+1} \right\} - 0, \\
&= \mathrm{ind}\, D_{\nabla_0^{\mathcal{E}}}.
\end{aligned}$$

Connes-Chern Character

Theorem (Connes)

1. *The cochain τ_{2k}^{D} is a cyclic cocycle.*
2. *The class of τ_{2k}^{D} in $\mathrm{HP}^0(\mathcal{A})$ is independent of the value of k.*

Definition

The class of τ_{2k}^{D} in $\mathrm{HP}^0(\mathcal{A})$ is denoted by $\mathrm{Ch}(D)$ and called the Connes-Chern character of $(\mathcal{A}, \mathcal{H}, D)$.

Theorem (Connes)

We have

$$\mathrm{ind}\, D_{\nabla^{\mathcal{E}}} = \langle \mathrm{Ch}(D), \mathrm{Ch}(\mathcal{E}) \rangle \quad \forall (\mathcal{E}, \nabla^{\mathcal{E}}).$$

Example: Spectral Triples over NC Tori

Theorem (Connes)

Let $(\mathcal{A}_\theta, \mathcal{H}, D)$ be Connes' spectral triple over the noncommutative torus $\mathcal{A}_\theta$ (cf. Part 2). Then

1. $(\mathcal{A}_\theta, \mathcal{H}, D)$ *is* 2^+*-summable.*
2. *Up to a constant multiple, the Connes-Chern character of* $(\mathcal{A}_\theta, \mathcal{H}, D)$ *is represented by the cyclic 2-cocycle,*

$$\tau_2(a^0, a^1, a^2) = \tau_0 \left[a^0 \left(\delta_1(a^1)\delta_2(a^2) - \delta_2(a^1)\delta_1(a^2) \right) \right],$$

   where τ_0 *is the canonical trace and the* δ_j *the basic derivations of* $\mathcal{A}_\theta$*.*

Remark

This Connes-Chern character plays an important role in Bellissard's work on the (integral) quantum Hall effect.

The JLO Cochain

Notation

For $t > 0$ and $X^0, \ldots, X^m$ in $\mathcal{L}(\mathcal{H})$, set

$$H_t(X^0, \ldots, X^m) = \int_{\Delta_m} X^0 e^{-s_0 t D^2} \cdots X^m e^{-s_m t D^2},$$

where $\Delta_m = \{s_j \geq 0; s_0 + s_1 + \cdots + s_m = 1\}$.

Definition (Jaffe-Lesniewski-Osterwalder)

For $t > 0$, the JLO cochain $\varphi^{\mathrm{JLO}} = (\varphi^{\mathrm{JLO}}_{t,2k})_{k \geq 0}$ is given by

$$\varphi^{\mathrm{JLO}}_{t,2k}(a^0, \ldots, a^{2k}) = t^q \operatorname{Str}\left\{ H_t\left(a^0, [D, a^1], \ldots, [D, a^{2k}] \right) \right\}, \ a^j \in \mathcal{A}.$$

The JLO Cochain

Proposition (Jaffe-Lesniewski-Osterwalder, Connes, Getzler-Szenes)

1. $(b+B)\varphi_t^{\mathrm{JLO}} = 0$ *for all* $t > 0$.
2. *Suppose that* $\mathcal{E} \simeq e\mathcal{A}^q$, $e^2 = e \in M_q(\mathcal{A})$. *Then*

$$\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \langle \varphi_t^{\mathrm{JLO}}, \mathrm{Ch}(e)\rangle \quad \forall t > 0.$$

Remarks

1. As it may have infinitely many components $\varphi_{t,2k}^{\mathrm{JLO}} \neq 0$, in general the JLO cochain is not an even periodic cyclic cocycle.
2. It is however a cocycle in "entire cyclic cohomology".
3. The JLO cochain can be interpreted as the Chern character of some superconnection on the space of chains (Quillen).

The CM Cocycle

Assumptions

1. $(\mathcal{A}, \mathcal{H}, D)$ is p^+-summable for some $p \geq 1$.
2. There are asymptotics in $t^\alpha(\log t)^\beta$ for $\operatorname{Str}\left\{H_t(X^0, \ldots, X^m)\right\}$ as $t \to 0^+$, for $X^j = a$, or $[D, a]$, or D.

Theorem (Connes-Moscovici)

Define $\varphi^{\mathrm{CM}} = (\varphi^{\mathrm{CM}}_{2k})_{k\geq 0}$ *by*

$$\varphi^{\mathrm{CM}}_{2k}(a^0, \ldots, a^m) = \operatorname{Pf}_{t\to 0+} \varphi^{\mathrm{JLO}}_{t,2k}(a^0, \ldots, a^{2k}), \quad a^j \in \mathcal{A},$$

where Pf *is the "partie finie" (finite part). Then*

1. $\varphi^{\mathrm{CM}}_{2q} = 0$ *for* $q > \frac{1}{2}p$ *and* $(b + B)\varphi^{\mathrm{CM}} = 0$*, and hence* φ^{CM} *is an even periodic cocycle.*
2. *The class of* φ^{CM} *in* $\mathrm{HP}^0(\mathcal{A})$ *agrees with* $\mathrm{Ch}(D)$.
3. *We have*

$$\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \langle \varphi^{\mathrm{CM}}, \mathrm{Ch}(\mathcal{E})\rangle \qquad \forall (\mathcal{E}, \nabla^{\mathcal{E}}).$$

Example: Dirac Spectral Triple

Proposition (Block-Fox, Connes-Moscovici, RP '03, RP+Wang '15)

For a Dirac spectral triple $(C^\infty(M), L^2(M, \not{S}), \not{D})$*, we have*

$$\varphi_{2k}^{\mathrm{CM}}(f^0, \ldots, f^{2k}) = \frac{(2i\pi)^{-n}}{(2k)!} \int_M f^0 df^1 \wedge \cdots \wedge df^{2k} \wedge \hat{A}(R^M),$$

That is, we have

$$\varphi^{\mathrm{CM}} = \varphi_C,$$

where $C = (2i\pi)^{-n}\hat{A}(R^M)^\wedge$ *is the Poincaré dual current of* $(2i\pi)^{-n}\hat{A}(R^M)$*.*

Example: Dirac Spectral Triple

Setup

1. E is a Hermitian vector bundle and $\mathcal{E} = C^\infty(M, E)$.
2. ∇^E is a connection on E with curvature ∇^E.

Consequence

We have

$$\operatorname{ind} \not{D}_{\nabla^E} = \operatorname{ind} D_{\nabla^{\mathcal{E}}} = \langle \mathrm{Ch}(\not{D}), \mathrm{Ch}(\mathcal{E}) \rangle = \langle \varphi^{\mathrm{CM}}, \alpha \left(\mathrm{Ch}(F^E \right) \rangle,$$

where α is the HKR map. Using the formula for φ^{CM} we get

$$\langle \varphi^{\mathrm{CM}}, \alpha \left(\mathrm{Ch}(F^E \right) \rangle = \langle \varphi_C, \alpha \left(\mathrm{Ch}(F^E \right) \rangle = \langle C, \mathrm{Ch}(F^E) \rangle.$$

Thus,

$$\operatorname{ind} \not{D}_{\nabla^E} = \langle C, \mathrm{Ch}(F^E) \rangle = (2i\pi)^{-n} \int_M \hat{A}(R^M) \wedge \mathrm{Ch}(F^E).$$

This is the Atiyah-Singer index formula!