Introduction to Noncommutative Geometry
Lecture 3: Cyclic Homology and Cohomology

Raphaël Ponge

Seoul National University & UC Berkeley
www.math.snu.ac.kr/∼ponge

UC Berkeley, April 15, 2015

Overview of Noncommutative Geometry

Classical	**NCG**
Riemannian Manifold (M, g)	Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$
Vector Bundle E over M	Projective Module $\mathcal{E}$ over $\mathcal{A}$ $\mathcal{E} = e\mathcal{A}^q, \;\; e \in M_q(\mathcal{A}), \; e^2 = e$
$\mathrm{ind}\, \not{D}_{\nabla^E}$	$\mathrm{ind}\, D_{\nabla^{\mathcal{E}}}$
de Rham Homology/Cohomology	Cyclic Cohomology/Homology
Atiyah-Singer Index Formula $\mathrm{ind}\, \not{D}_{\nabla^E} = \int \hat{A}(R^M) \wedge \mathrm{Ch}(F^E)$	Connes-Chern Character $\mathrm{Ch}(D)$ $\mathrm{ind}\, D_{\nabla^{\mathcal{E}}} = \langle \mathrm{Ch}(D), \mathrm{Ch}(\mathcal{E}) \rangle$
Local Index Theorem	CM cocycle

Hochschild Homology

Setup

- $\mathcal{A}$ is a unital algebra over $\mathbb{C}$.

Definition

The Hoschild homology $\mathrm{HH}_\bullet(\mathcal{A})$ is the homology of the chain-complex $(C_\bullet(\mathcal{A}), b)$, where:

- The space of m-chains is $C_m(\mathcal{A}) = \mathcal{A}^{\otimes(m+1)}$.
- The boundary operator $b : C_m(\mathcal{A}) \rightarrow C_{m-1}(\mathcal{A})$ is given by

$$b(a^0 \otimes \cdots a^m) = \sum_{0\leq j\leq m-1} (-1)^j a^0 \otimes \cdots \otimes a^j a^{j+1} \otimes \cdots \otimes a^m \\ + (-1)^m a^m a^0 \otimes a^1 \otimes \cdots \otimes a^{m-1}.$$

Definition

The Hoschild cohomology $\mathrm{HH}^\bullet(\mathcal{A})$ is the cohomology of the cochain-complex $(C^\bullet(\mathcal{A}), b)$, where:

- The space of m-cochains is

$$C^m(\mathcal{A}) = \{(m+1)\text{-linear forms } \varphi : \mathcal{A}^{m+1} \to \mathbb{C}\} \simeq \left(\mathcal{A}^{\otimes(m+1)}\right)^*.$$

- The boundary operator $b : C^m(\mathcal{A}) \to C^{m+1}(\mathcal{A})$ is given by

$$(b\varphi)(a^0, \ldots, a^{m+1}) = \sum_{0\leq j\leq m} (-1)^j \varphi(a^0, \ldots, a^j a^{j+1}, \ldots, a^m) + (-1)^{m+1}\varphi(a^{m+1}a^0, a^1, \ldots, a^m).$$

Duality between $\mathrm{HH}^\bullet(\mathcal{A})$ and $\mathrm{HH}_\bullet(\mathcal{A})$

Proposition

There is a natural duality pairing $\langle\cdot,\cdot\rangle : C^m(\mathcal{A}) \times C_m(\mathcal{A}) \to \mathbb{C}$ *given by*

$$\langle \varphi, a^0 \otimes \cdots \otimes a^m \rangle = \varphi(a^0, \ldots, a^m).$$

It descends to a duality pairing,

$$\langle\cdot,\cdot\rangle : \mathrm{HH}^\bullet(\mathcal{A}) \times \mathrm{HH}_\bullet(\mathcal{A}) \longrightarrow \mathbb{C}.$$

Example: $\mathcal{A} = C^\infty(M)$

Setup

- $\mathcal{A} = C^\infty(M)$, M compact manifold.
- $\Omega^m(M) = C^\infty(M, \Lambda^m T^*_{\mathbb{C}} M)$.
- $\Omega_m(M) = \Omega^m(M)'$ (m-dimensional de Rham currents).

Theorem (Hochschild-Kostant-Rosenberg, Connes)

- *For $C \in \Omega_m(M)$ define $\varphi_C \in C^m(\mathcal{A})$ by*

$$\varphi_C(f^0, f^1, \ldots, f^m) = \langle C, f^0 df^1 \wedge \cdots \wedge df^m \rangle.$$

  Then φ_C is a Hochschild cochain.

- *If we restrict it to continuous cochains, then the map $C \to \varphi_C$ descends to an isomorphism,*

$$\Omega_\bullet(M) \simeq \mathrm{HH}^\bullet\left(C^\infty(M)\right).$$

Example: $\mathcal{A} = C^\infty(M)$

Definition (Hochschild-Kostant-Rosenberg)

Define $\alpha : C_m(\mathcal{A}) \to \Omega^m(\mathcal{A})$ is by

$$\alpha(f^0 \otimes f^1 \otimes \cdots \otimes f^m) = \frac{1}{m!} f^0 df^1 \wedge \cdots \wedge df^m.$$

This is called the Hochschild-Kostant-Rosenberg map.

Theorem (Hochschild-Kostant-Rosenberg, Connes)

1. *The HKR map is a morphism of complexes from $(C_\bullet(\mathcal{A}), b)$ to $(\Omega^\bullet(M), 0)$.*
2. *If we define the Hoschild homology by using the topological tensor product $\hat{\otimes}$, then α induces an isomorphism,*

$$\mathrm{HH}_\bullet(C^\infty(M)) \simeq \Omega^\bullet(M).$$

Cyclic Cohomology (Connes, Tsygan)

Definition

A cochain $\varphi \in C^m(\mathcal{A})$ is cyclic when

$$\varphi(a^m, a^0, \ldots, a^{m-1}) = (-1)^m \varphi(a^0, \ldots, a^m) \qquad \forall a^j \in \mathcal{A}.$$

We denote by $C^m_\lambda(\mathcal{A})$ the space of cyclic m-cochains.

Lemma

The Hochschild coboundary preserves the cyclic condition.

Definition

The cyclic cohomology $H^\bullet_\lambda(\mathcal{A})$ is the cohomology of the sub-complex $(C^\bullet_\lambda(\mathcal{A}), b)$.

Definition

The cyclic operator $T : C_m(\mathcal{A}) \to C_m(\mathcal{A})$ is defined by

$$T(a^0 \otimes \cdots \otimes a^m) = (-1)^m a^m \otimes a^0 \otimes \cdots \otimes a^{m-1}, \qquad a^j \in \mathcal{A}.$$

The space of co-cyclic m-chains is $C_m^\lambda(\mathcal{A}) := C_m(\mathcal{A})/\operatorname{ran}(1-T)$.

Lemma

The Hoschschild boundary preserves $\operatorname{ran}(1-T)$, *and hence descends to*

$$b : C_m^\lambda(\mathcal{A}) \longrightarrow C_{m-1}^\lambda(\mathcal{A}).$$

Definition

The cyclic homology $H_\bullet^\lambda(\mathcal{A})$ is the homology of the chain-complex $\big(C_\bullet^\lambda(\mathcal{A}), b\big)$.

Connes' B-Operator

Definition (Connes)

The operator $B : C_m(\mathcal{A}) \to C_{m+1}(\mathcal{A})$ is the composition,

$$B = (1 - T)B_0 A, \quad A = 1 + T + \cdots + T^m,$$

where $B_0 : C_m(\mathcal{A}) \to C_{m+1}(\mathcal{A})$ is given by

$$B_0(a^0 \otimes \cdots \otimes a^m) = 1 \otimes a^0 \otimes \cdots \otimes a^m.$$

Lemma (Connes)

We have

$$B^2 = 0 \qquad \text{and} \qquad bB + Bb = 0.$$

The (b, B)-Mixed Complex

Consequence

We have a mixed chain-complex,

$$C_{\bullet-1}(\mathcal{A}) \xleftarrow{b} C_\bullet(\mathcal{A}) \xrightarrow{B} C_{\bullet+1}(\mathcal{A}), \quad b^2 = B^2 = bB + Bb = 0.$$

Definition

$\mathrm{HC}_\bullet(\mathcal{A})$ is the homology of this mixed complex,i.e., the homology of the chain-complex $(\mathcal{B}_\bullet(\mathcal{A}), b + B)$, where

$$\mathcal{B}_m(\mathcal{A}) := \bigoplus_{p+q=m} C_{p-q}(A) = C_m(\mathcal{A}) \oplus C_{m-2}(\mathcal{A}) \oplus \cdots .$$

Remark

$\mathrm{HC}_\bullet(\mathcal{A})$ is the homology of the total complex of the bicomplex,

$$(\mathcal{B}_{\bullet,\bullet}(\mathcal{A}), b, B), \quad \text{where } \mathcal{B}_{p,q}(\mathcal{A}) = C_{p-q}(\mathcal{A}),$$

with the convention that $C_{p-q}(\mathcal{A}) = \{0\}$ when $p - q \leq 0$.

The (b, B)-Description of Cyclic Homology

Proposition (Connes)

The canonical projection,

$$\mathcal{B}_m(\mathcal{A}) \longrightarrow C_m(\mathcal{A}) \longrightarrow C_m^\lambda(\mathcal{A}) = C_m(\mathcal{A})/\operatorname{ran}(1-T)$$

induces an isomorphism,

$$HC_\bullet(\mathcal{A}) \simeq H_\bullet^\lambda(\mathcal{A}).$$

Example: $\mathcal{A} = C^\infty(M)$

Theorem (Connes)

1. *The HKR map* $\alpha : C_\bullet(\mathcal{A}) \to \Omega^\bullet(M)$ *is a morphism of mixed complexes from* $(C_\bullet(\mathcal{A}), b, B)$ *to* $(\Omega^\bullet(M), 0, d)$ *(where* d *is the de Rham boundary).*
2. *If we define* $C_\bullet(\mathcal{A})$ *by using the topological tensor product, then the HKR map induces isomorphisms,*

$$HC_m\left(C^\infty(M)\right) \simeq H_m(M) \oplus H_{m-2}(M) \oplus \cdots ,$$

where $H_\bullet(M)$ *is the de Rham cohomology.*

The Chern Character

Setup

- E is a vector bundle over a manifold M
- ∇^E is a connection over E with curvature F^E.

Definition

The Chern form of F^E is

$$\mathrm{Ch}(F^E) := \mathrm{Tr}\left[\exp(-F^E)\right] \in \Omega^{\mathrm{ev}}(M),$$

where $\Omega^{\mathrm{ev}}(M) := \Omega^0(M) \oplus \Omega^2(M) \oplus \cdots$.

Theorem (Chern, Weil)

1. $\mathrm{Ch}(F^E)$ *is a closed form.*
2. *Its class in* $H^{\mathrm{ev}}(M) := H^0(M) \oplus H^2(M) \oplus \cdots$ *does not depend on the choice of* ∇^E.

Periodic Cyclic Homology

Definition

The periodic cyclic homology $\mathrm{HP}_\bullet(\mathcal{A})$ is the homology of the chain-complex,

$$C_{[0]}(\mathcal{A}) \overset{b+B}{\rightleftarrows} C_{[1]}(\mathcal{A}), \quad \text{where } C_{[i]}(\mathcal{A}) = \prod_{q\geq 0} C_{2q+i}(\mathcal{A}).$$

Remarks

1. An even periodic cyclic cycle is an infinite sequence $\omega = (\omega_{2q})_{q\geq 0}$, $\omega \in C_2 q(\mathcal{A})$, such that
$$b\omega_{2q} + B\omega_{2q-2} = 0 \quad \forall q \geq 1.$$
2. There is a similar description of odd periodic cyclic cycles.

Example: $\mathcal{A} = C^\infty(M)$

Theorem (Connes)

Let M be a compact manifold. If we define $C_\bullet(\mathcal{A})$ by using the topological tensor product, then the HKR map induces isomorphisms,

$$\mathrm{HP}_0\left(C^\infty(M)\right) \simeq H^{\mathrm{ev}}(M) \quad \textit{and} \quad \mathrm{HP}_1\left(C^\infty(M)\right) \simeq H^{\mathrm{odd}}(M).$$

Chern Character Revisited

Setup

- $\mathcal{A} = C^\infty(M)$ and $e^2 = e \in M_q(\mathcal{A}) = C^\infty(M, M_q(\mathbb{C}))$.
- $E = \sqcup_{x \in M} \operatorname{ran} e(x) \subset M \times \mathbb{C}^q$.
- ∇_0^E is the Grassmannian connection, i.e.,

$$C^\infty(M, E) \xrightarrow{d} C^\infty(M, T^*M \otimes \mathbb{C}^q) \xrightarrow{1 \otimes e} C^\infty(M, E).$$

Lemma

Let F_0^E be the curvature of ∇_0^E. Then

$$F_0^E = e(de)^2 e = e(de)^2,$$

$$\mathrm{Ch}(F_0^E) = \sum_{k \geq 1} \frac{(-1)^k}{k!} \operatorname{Tr}\left[e(de)^{2k}\right].$$

Corollary

In terms of the HKR map $\alpha : C_\bullet(\mathcal{A}) \to \Omega^\bullet(M)$, *we have*

$$\mathrm{Ch}(F_0^E) = \alpha\left(\mathrm{Ch}^0(e)\right), \qquad \mathrm{Ch}^0(e) = \left(\mathrm{Ch}^0_{2k}(e)\right)_{k\geq 0},$$

where

$$\mathrm{Ch}^0_{2k}(e) = (-1)^k \frac{(2k)!}{k!} \,\mathrm{Tr}\left[\overbrace{e \otimes e \otimes \cdots \otimes e}^{(2k+1) \text{ times}}\right], \quad k \geq 0.$$

The Chern Character in Cyclic Homology

Setup

- $\mathcal{A}$ is a unital $\mathbb{C}$-algebra.
- $\mathcal{E} \simeq e\mathcal{A}^q$, $e^2 = e \in M_q(\mathcal{A})$, is a f.g. projective module.

Definition

The Chern character of e is the even periodic cyclic chain,

$$\mathrm{Ch}(e) = (\mathrm{Ch}_{2k}(e))_{k \geq 0} \in C_{[0]}(\mathcal{A}),$$

where

$$\mathrm{Ch}_0(e) = \mathrm{Tr}\left[e\right],$$

$$\mathrm{Ch}_{2k}(e) = (-1)^k \frac{(2k)!}{k!} \mathrm{Tr}\left[(e - \frac{1}{2}) \otimes \overbrace{e \otimes \cdots \otimes e}^{2k \text{ times}}\right], \quad k \geq 1.$$

The Chern Character in Cyclic Homology

Theorem (Connes, Getzler-Szenes)

1. $\operatorname{Ch}(e)$ *is a periodic cyclic cycle, i.e.,* $(b+B)\operatorname{Ch}(e)=0$.
2. *The class of* $\operatorname{Ch}(e)$ *in* $\operatorname{HP}_\bullet(\mathcal{A})$ *depends only on* $\mathcal{E}$, *and not on the choice of* e *such that* $\mathcal{E}\simeq e\mathcal{A}^q$.

Definition

The class of $\operatorname{Ch}(e)$ in $\operatorname{HP}_\bullet(\mathcal{A})$ is denoted by $\operatorname{Ch}(\mathcal{E})$ and is called the Chern character of $\mathcal{E}$.

Cyclic Homology of $\mathcal{A} = \mathbb{C}G$

Setup

- $\mathcal{A} = \mathbb{C}G$, where G is a (discrete) group.

Definition

The group homology $H_\bullet(G)$ is the homology of the complex $(C_\bullet(G), \delta)$, where

- $C_m(G) = \mathbb{C}[G^m] = \text{Span}\{(g_1, \ldots, g_m);\ g_j \in G\}$.
- $\delta : C_m(G) \to C_{m-1}(G)$ is given by

$$\delta(g_1, \ldots, g_m) = \sum_{j=1}^{m-1} (-1)^j (g_1, \ldots, g_j g_{j+1}, \ldots, g_m) + (-1)^m (g_1, \ldots, g_{m-1}).$$

Remark

$H_\bullet(G) \simeq H_\bullet(BG)$, where BG is the classifying space of G.

Cyclic Homology of $\mathcal{A} = \mathbb{C}G$

Lemma

Let $h \in G$. For $m \in \mathbb{N}_0$ define

$$C_m(\mathbb{C}G)_h = \text{Span}\left\{(g_1, \ldots, g_m);\ g_1 \cdots g_m \in [h]\right\},$$

where $[h]$ is the conjugation class of h. Then

$$b\left(C_m(\mathbb{C}G)_h\right) \subset C_{m-1}(\mathbb{C}G)_h \text{ and } B\left(C_m(\mathbb{C}G)_h\right) \subset C_{m+1}(\mathbb{C}G)_h.$$

Definition

$\mathrm{HC}_\bullet(\mathbb{C}G)_h$ is the homology of the sub-mixed-complex $(C_\bullet(\mathbb{C}G), b, B)$.

Proposition

We have

$$\mathrm{HC}_\bullet(\mathbb{C}G) = \bigoplus_{[h]} \mathrm{HC}_\bullet(\mathbb{C}G)_h.$$

where $[h]$ ranges over the conjugation classes of G.

Cyclic Homology of $\mathcal{A} = \mathbb{C}G$

Notation

- $Z_h = \{g \in G;\ gh = hg\}$ is the centralizer of $h \in G$.
- $N_h = Z_h / {<}h{>}$ is the normalizer of h (where ${<}h{>}$ is the sugroup generated by h).

Theorem (Burghelea)

1. *If h is torsion (i.e., $h^r = 1$ for some $r \geq 1$), then*

$$\begin{aligned} \mathrm{HC}_m(\mathbb{C}G)_h &\simeq H_m(Z_h) \oplus H_{m-2}(Z_h) \oplus \cdots, \\ &\simeq H_m(N_h) \oplus H_{m-2}(N_h) \oplus \cdots. \end{aligned}$$

2. *If h is not torsion, then*

$$\mathrm{HC}_m(\mathbb{C}G)_h \simeq H_m(N_h).$$