Introduction to Noncommutative Geometry
Part 2: Spectral Triples

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UC Berkeley, April 1 & 8, 2015

Overview of Noncommutative Geometry

Classical	**NCG**
Riemannian Manifold (M, g)	Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$
Vector Bundle E over M	Projective Module $\mathcal{E}$ over $\mathcal{A}$ $\mathcal{E} = e\mathcal{A}^q, \;\; e \in M_q(\mathcal{A}), \; e^2 = e$
$\mathrm{ind} \not{D}_{\nabla^E}$	$\mathrm{ind}\, D_{\nabla^\mathcal{E}}$
de Rham Homology/Cohomology	Cyclic Cohomology/Homology
Atiyah-Singer Index Formula $\mathrm{ind} \not{D}_{\nabla^E} = \int \hat{A}(R^M) \wedge \mathrm{Ch}(F^E)$	Connes-Chern Character $\mathrm{Ch}(D)$ $\mathrm{ind}\, D_{\nabla^\mathcal{E}} = \langle \mathrm{Ch}(D), \mathrm{Ch}(\mathcal{E}) \rangle$
Local Index Theorem	CM cocycle

Spectral Triples

Definition (Connes-Moscovici)

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of

1. A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
2. A $*$-algebra $\mathcal{A}$ represented in $\mathcal{H}$.
3. A selfadjoint unbounded operator D on $\mathcal{H}$ such that
    1. D maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
    2. $(D \pm i)^{-1}$ is compact.
    3. $[D, a]$ is bounded for all $a \in \mathcal{A}$.

Remark

When $\mathcal{H}^- \neq \{0\}$, we say that $(\mathcal{A}, \mathcal{H}, D)$ is an even spectral triple. Otherwise we say that $(\mathcal{A}, \mathcal{H}, D)$ is an odd spectral triple

Dirac Spectral Triple

Example

- (M^n, g) compact Riemannian spin manifold (n even) with spinor bundle $\not{S} = \not{S}^+ \oplus \not{S}^-$.
- $\not{D}_g : C^\infty(M, \not{S}) \to C^\infty(M, \not{S})$ is the Dirac operator of (M, g).
- $C^\infty(M)$ acts by multiplication on $L^2_g(M, \not{S})$.

Then $\left(C^\infty(M), L^2\left(M, \not{S}\right), \not{D}_g \right)$ is a spectral triple.

Remark

We also get spectral triples by taking

- $\mathcal{H} = L^2(M, \Lambda^\bullet T^*M)$ and $D = d + d^*$.
- $\mathcal{H} = L^2(M, \Lambda^{0,\bullet} T^*_{\mathbb{C}} M)$ and $D = \overline{\partial} + \overline{\partial}^*$ (when M is a complex manifold).

Metrics from Spectral Triples

Proposition (Connes)

Let $d(x,y)$ be the Riemannian distance of (M,g). Then

$$d(x,y) = \inf\left\{|f(x)-f(y)|;\ \|[\not{D}_g, f]\| \leq 1\right\} \quad \forall x,y \in M.$$

Remark

Given a general spectral triple, we get a metric on the space of states of $\mathcal{A}$,

$$d(\varphi,\psi) := \inf\left\{|\varphi(a)-\psi(a)|;\ \|[D,a]\| \leq 1\right\}.$$

The above formulas were a main impetus for Rieffel's quantum metric spaces.

Noncommutative Torus

Setup

- Given $\theta \in [0,1)$, we let $\mathbb{Z}$ acts on $\mathbb{T} = \mathbb{S}^1$ by

$$k \cdot z = e^{2i\pi\theta k} z, \quad z \in \mathbb{T},\ k \in \mathbb{Z}.$$

This is the action generated by the rotation of angle $2\pi\theta$.

Fact

When $\theta \notin \mathbb{Q}$, the orbits of the action are dense. (In this case $\theta\mathbb{Z} + 2\pi\mathbb{Z}$ is a dense subgroup of $\mathbb{R}$.)

Example

Plots of orbit points $z_k = e^{2i\pi\theta k} \cdot 1$ with $\theta = 1/2\pi$ and $k = 0, \ldots, p$ for increasing values of $p = 5, 10, 20, 50, 100, 150, \ldots$.

Crossed-Product Algebra $C^\infty(\mathbb{T}) \rtimes_\theta \mathbb{Z}$

Definition

$C^\infty(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ is $C^\infty(\mathbb{T}) \otimes \mathbb{Z}$ with product and involution,

$$(f_1 \otimes k_1)(f_2 \otimes k_2) = f_1(k_1 \cdot f_2) \otimes (k_1 + k_2),$$
$$(f \otimes k)^* = -\overline{f} \otimes k,$$

where $(k \cdot f)(z) = f(e^{-2i\pi\theta} z)$.

Lemma (Fourier Series Decomposition in $C^\infty(\mathbb{T})$)

Let $f(z) = \sum_{m\in\mathbb{Z}} a_m z^m \in L^2(\mathbb{T})$. Then TFAE:

1. $f(z) \in C^\infty(\mathbb{T})$.
2. $(a_m) \in \mathcal{S}(\mathbb{Z})$,

where $\mathcal{S}(\mathbb{Z}) := \left\{(a_m)_{m\in\mathbb{Z}} \subset \mathbb{C};\ |a_m| = \mathrm{O}\left(|m|^{-N}\right)\ \forall N \geq 1\right\}$.

Crossed-Product Algebra $C^\infty(\mathbb{T}) \rtimes_\theta \mathbb{Z}$

Proposition

Define operators U and V of $L^2(\mathbb{T})$ by

$$(U\xi)(z) = z\xi(z) \quad \text{and} \quad (V\xi)(z) = \xi(e^{-2i\pi\theta}z) \quad \forall \xi \in L^2(\mathbb{T}).$$

1. *U and V are unitary operators such that*

$$VU = e^{-2i\pi\theta}UV.$$

2. *The map $f \otimes k \to f(U)V^k$ yields an algebra isomorphism,*

$$C^\infty(\mathbb{T}) \rtimes_\theta \mathbb{Z} \simeq \left\{ \sum_{m\in\mathbb{Z}} \sum_{|k|<N} a_{m,k} U^m V^k ; (a_{m,k}) \in \mathcal{S}(\mathbb{Z})\ \forall k \right\}.$$

The Noncommutative Torus

Definition

The noncommutative torus is the algebra,

$$\mathcal{A}_\theta = \left\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n; (a_{m,n}) \in \mathcal{S}(\mathbb{Z}^2) \right\} \subset \mathcal{L}\left(L^2(\mathbb{T})\right).$$

Remarks

1. $\mathcal{A}_\theta$ contains the crossed-product algebras $C^\infty(\mathbb{T}) \rtimes_\theta \mathbb{Z}$.
2. The closure of $\mathcal{A}_\theta$ in $\mathcal{L}\left(L^2(\mathbb{T})\right)$ is also called noncommutative torus. (This is a C^*-algebra.)

Example

For $\theta = 0$, we have the algebra isomorphism,

$$\mathcal{A}_\theta \ni \sum a_{m,n} U^m V^n \longrightarrow \sum a_{m,n} z^m w^n \in C^\infty(\mathbb{T}^2).$$

Canonical Trace

Proposition

Define $\tau_0 : \mathcal{A}_\theta \to \mathbb{C}$ *by*

$$\tau_0\left(\sum a_{m,m} U^m V^n\right) = a_{00}.$$

Then τ_0 *is the unique trace on* $\mathcal{A}_\theta$ *such that* $\tau_0(1) = 1$.

Remark

If $e \in \mathcal{A}_\theta$ is a Powers-Rieffel idempotent, then $\tau_0(e) = \theta$. Thus,

$$\mathcal{A}_\theta \not\simeq \mathcal{A}_{\theta'} \quad \text{when } \theta \neq \theta'.$$

The Basic Derivations

Proposition

For $j = 1, 2$ *define* $\delta_j : \mathcal{A}_\theta \to \mathcal{A}_\theta$ *by*

$$\delta_1(U^m V^n) = mU^m V^n \quad \textit{and} \quad \delta_2(U^m V^n) = nU^m V^n.$$

Then δ_1 *and* δ_2 *are derivations of the algebra* $\mathcal{A}_\theta$*, i.e.,*

$$\delta_j(ab) = \delta_j(a)b + a\delta_j(b) \qquad \forall a, b \in \mathcal{A}_\theta.$$

Remarks

1. δ_1 and δ_2 are called the basic derivations of $\mathcal{A}_\theta$.
2. For $\theta = 0$, under

$$\mathcal{A}_0 \simeq C^\infty(\mathbb{T}^2) \simeq \left\{ \sum a_{m,n} e^{2im\pi x} e^{2im\pi y} \right\},$$

the derivations δ_1 and δ_2 correspond to

$$(2i\pi)^{-1} \frac{\partial}{\partial x} \qquad \text{and} \qquad (2i\pi)^{-1} \frac{\partial}{\partial y}.$$

Holomorphic Structures on $\mathcal{A}_\theta$

Fact

Up to the action of the modular group $\mathrm{PSL}(2,\mathbb{Z})$*, the holomorphic structures on* $\mathbb{T}^2$ *are parametrized by complex numbers* τ*,* $\Im\tau > 0$*, and the associate holomorphic differentials,*

$$\partial_Z = \partial_x + \overline{\tau}^{-1}\partial_y, \qquad Z = (2\pi)^{-1}(x + \tau y).$$

Definition

A holomorphic structure on $\mathcal{A}_\theta$ is given by $\tau \in \mathbb{C}$, $\Im\tau > 0$, and the associate holomorphic derivation,

$$\partial = \delta_1 + \overline{\tau}\delta_2.$$

Remark

In what follows we shall take $\tau = i = \sqrt{-1}$.

The Hilbert Space $\mathcal{H}_0$

Lemma

The canonical trace τ_0 defines an inner-product on $\mathcal{A}_\theta$ by

$$\langle a, b\rangle_0 := \tau_0\left(b^* a\right) \qquad \forall a, b \in \mathcal{A}_\theta.$$

Remark

The family $\{U^m V^n\}$ is orthonormal with respect to $\langle \cdot, \cdot \rangle_0$.

Definition

The Hilbert space $\mathcal{H}_0$ is the completion of $\mathcal{A}_\theta$ with respect to $\langle \cdot, \cdot \rangle_0$.

Remarks

1. The algebra $\mathcal{A}_\theta$ acts on $\mathcal{H}_0$ by left-multiplication (left-regular representation).
2. There is also a right-action (right-regular representation), and so $\mathcal{H}$ is an $\mathcal{A}_\theta$-bimodule.

The Hilbert space $\mathcal{H}^{1,0}$

Definition

1. The space of holomorphic 1-forms is

$$\mathcal{A}_\theta^{1,0} = \mathrm{Span}\left\{a\partial(b);\ a, b \in \mathcal{A}_\theta\right\}.$$

2. $\mathcal{H}^{1,0}$ is the completion of $\mathcal{A}_\theta$ w.r.t. $\langle \cdot, \cdot \rangle_0$.

Remarks

1. $\mathcal{A}_\theta^{1,0}$ is an $\mathcal{A}_\theta$-bimodule, and so is $\mathcal{H}^{1,0}$.
2. The derivation ∂ maps $\mathcal{A}_\theta$ to $\mathcal{A}_\theta^{1,0}$. We shall also denote by ∂ its closure as an unbounded operator from $\mathcal{H}_0$ to $\mathcal{H}^{1,0}$.

Spectral Triples over NC Tori

Theorem (Connes)

The triple $(\mathcal{A}_\theta, \mathcal{H}, D)$ *is a spectral triple, where*

- $\mathcal{H}$ *is the Hilbert space* $\mathcal{H}_0 \oplus \mathcal{H}^{1,0}$.
- $\mathcal{A}_\theta$ *is represented in* $\mathcal{H}$ *by left-multiplication operators.*
- D *is the unbounded operator of* $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}^{1,0}$ *to itself given by*

$$D = \begin{pmatrix} 0 & \partial^* \\ \partial & 0 \end{pmatrix},$$

  where ∂^* *is the adjoint of* ∂.

Remark

The operator D is isospectral to the operator $\partial + \partial^*$ on the ordinary torus $\mathbb{T}^2$.

Spectral Triples over NC Tori

Definition

The opposite algebra $\mathcal{A}_\theta^\circ$ has same underlying vector space structure as $\mathcal{A}_\theta$ and opposite product,

$$a \cdot^\circ b := ba \qquad \forall a, b \in \mathcal{A}_\theta^\circ.$$

Remark

The right-actions of $\mathcal{A}_\theta$ on $\mathcal{H}_0$ and $\mathcal{H}^{1,0}$ give rise to left-actions of $\mathcal{A}_\theta^\circ$. Therefore, we may represent $\mathcal{A}_\theta^\circ$ by right-multiplication operators on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}^{1,0}$.

Theorem (Connes)

The triple $(\mathcal{A}_\theta^\circ, \mathcal{H}, D)$ is a spectral triple as well.

Remark

The spectral triples $(\mathcal{A}_\theta, \mathcal{H}, D)$ and $\left(\mathcal{A}_\theta^{\text{op}}, \mathcal{H}, D\right)$ satisfy some form of Poincaré duality in NCG (*cf.* Connes' book).

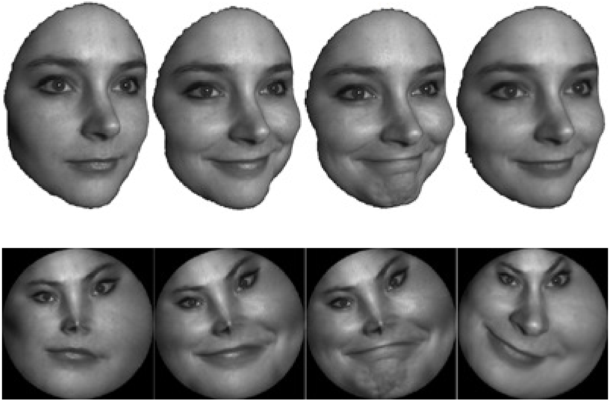
Conformal Geometry

Definition

1. Conformal geometry is the geometry up to angle-preserving transformations.
2. Two metrics g_1 and g_2 are conformally equivalent when

$$g_2 = k^{-2} g_1 \quad \text{for some } k \in C^\infty(M),\ k > 0.$$

Conformal Geometry



Conformal Changes of Metrics

Setup

- $(C^\infty(M), L^2_g(M, \$), \not{D}_g)$ is a Dirac spectral triple.
- Conformal change of metric: $\hat{g} = k^{-2}g$, $k \in C^\infty(M)$, $k > 0$.

Observation

Define $U : L^2_g(M, \$) \to L^2_{\hat{g}}(M, \$)$ be defined by

$$Uf = k^{\frac{n}{2}}\xi \quad \forall f \in L^2_g(M, \$).$$

Then U is a unitary operator and intertwines the spectral triples

$$\left(C^\infty(M), L^2_{\hat{g}}(M, \$), \not{D}_{\hat{g}}\right) \quad \text{and} \quad \left(C^\infty(M), L^2_g(M, \$), \sqrt{k}\not{D}_g\sqrt{k}\right).$$

In particular,

$$U\not{D}_{\hat{g}}U^* = \sqrt{k}\not{D}_g\sqrt{k}.$$

Twisted Spectral Triples

Definition (Connes-Moscovici)

A twisted spectral triple $(\mathcal{A}, \mathcal{H}, D)$ $(\mathcal{A}, \mathcal{H}, D)_\sigma$ consists of

1. A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
2. An involutive algebra $\mathcal{A}$ represented in $\mathcal{H}$ together with an automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ such that $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in \mathcal{A}$.
3. A selfadjoint unbounded operator D on $\mathcal{H}$ such that
    1. D maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
    2. $(D \pm i)^{-1}$ is compact.
    3. $[D, a]_\sigma := Da - \sigma(a)D$ is bounded for all $a \in \mathcal{A}$.

Theorem (Connes-Moscovici)

Consider the following:

- *An ordinary spectral triple* $(\mathcal{A}, \mathcal{H}, D)$.
- *A positive element* $k \in \mathcal{A}$ *with associated inner automorphism* $\sigma(a) = k^2 a k^{-2}$, $a \in \mathcal{A}$.

Then $(\mathcal{A}, \mathcal{H}, kDk)_\sigma$ *is a twisted spectral triple.*

Pseudo-Inner Twistings (RP+H. Wang)

Theorem (RP+H. Wang '15)

Consider the following:

- *An ordinary spectral triple* $(\mathcal{A}, \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-, D)$.
- *Positive elements* $k^\pm \in \mathcal{A}$ *with* $k^+k^- = k^-k^+$ *and associated inner automorphisms* $\sigma^\pm(a) = k^\pm a (k^\pm)^{-1}$.
- *A positive even operator* $\omega = \begin{pmatrix} \omega^+ & 0 \\ 0 & \omega^- \end{pmatrix} \in \mathcal{L}(\mathcal{H})$ *such that*

$$\omega^\pm a = \sigma^\pm(a)\omega^\pm \qquad \forall a \in \mathcal{A}.$$

Set $k = k^+k^-$ *and* $\sigma(a) = kak^{-1}$. *Then* $(\mathcal{A}, \mathcal{H}, \omega D \omega)_\sigma$ *is a twisted spectral triple.*

Conformal Dirac Spectral Triple

Setup

1. M^n compact spin (oriented) manifold (n even).
2. $\mathcal{C}$ is a conformal structure on M, i.e., a conformal class of metrics.
3. G is a group of diffeomorphisms preserving $\mathcal{C}$ and the spin structure. Thus, given any metric $g \in \mathcal{C}$ and $\phi \in G$,

$$\phi_* g = k_\phi^{-2} g \text{ with } k_\phi \in C^\infty(M), \ k_\phi > 0.$$

4. $C^\infty(M) \rtimes G$ crossed-product algebra, i.e., $C^\infty(M) \otimes \mathbb{C}G$ with product and involution,

$$(f_1 \otimes \phi_1)(f_2 \otimes \phi_2) = f_1(f_2 \circ \phi_1^{-1}) \otimes \phi_1 \phi_2,$$
$$(f \otimes \phi)^* = \overline{f} \otimes \phi^{-1}.$$

Conformal Dirac Spectral Triple

Lemma (Connes-Moscovici '08)

For $\phi \in G$ define $U_\phi : L^2_g(M, \$) \to L^2_g(M, \$)$ by

$$U_\phi \xi = k_\phi^{-\frac{n}{2}} \phi_* \xi \quad \forall \xi \in L^2_g(M, \$).$$

Then U_ϕ is a unitary operator, and

$$U_\phi \not{D}_g U_\phi^* = \sqrt{k_\phi} \not{D}_g \sqrt{k_\phi}.$$

Theorem (Connes-Moscovici '08)

The datum of any metric $g \in \mathcal{C}$ defines a twisted spectral triple $\left(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g \right)_{\sigma_g}$ given by

1. *The Dirac operator $\not{D}_g$ associated with g.*
2. *The representation $f \otimes \phi \to fU_\phi$ of $C^\infty(M) \rtimes G$ in $L^2_g(M, \$)$.*
3. *The automorphism $\sigma_g(fU_\phi) := k_\phi^{-1} fU_\phi$.*

Conformal Weights on NC Tori

Definition

A conformal weight on $\mathcal{A}_\theta$ is of the form,

$$\varphi(a) = \tau_0(ak^{-2}), \qquad k \in \mathcal{A}_\theta, \ k > 0.$$

We call k the Weyl factor of φ.

Fact

A conformal weight defines an inner product on $\mathcal{A}_\theta$ by

$$\langle a, b \rangle_\varphi := \varphi(b^* a) = \tau_0(b^* a k^{-2}), \quad a, b \in \mathcal{A}_\theta.$$

Twisted Spectral Triples on NC Tori

Theorem (Connes-Tretkoff)

Consider the following:

- *The Hilbert space* $\mathcal{H}_\varphi := \mathcal{H}^0_\varphi \oplus \mathcal{H}^{1,0}$*, where* $\mathcal{H}^0_\varphi$ *is the completion of* $\mathcal{A}_\theta$ *with respect to* $\langle \cdot, \cdot \rangle_\varphi$*.*
- *The operator* $D_\varphi := \begin{pmatrix} 0 & \partial^*_\varphi \\ \partial & 0 \end{pmatrix}$*, where* ∂^*_φ *is the adjoint of* ∂ *with respect to* $\langle \cdot, \cdot \rangle_\varphi$*.*
- *The representation* $a \to \begin{pmatrix} (k^{-1}ak)^\circ & 0 \\ 0 & a^\circ \end{pmatrix}$ *of* $\mathcal{A}$ *in* $\mathcal{H}_\varphi$*, where* $^\circ$ *denotes the right-action.*
- *The inner automorphism* $\sigma(a) = k^{-1}ak$ *of* $\mathcal{A}_\theta$*.*

Then $(\mathcal{A}^\circ_\theta, \mathcal{H}_\varphi, D_\varphi)_\sigma$ *is a twisted spectral triple.*

Remark

$(\mathcal{A}_\theta, \mathcal{H}_\varphi, D_\varphi)$ is an ordinary spectral triple.

Twisted Spectral Triples on NC Tori

Lemma (Connes-Tretkoff)

The right-multiplication by k on $\mathcal{A}_\theta$ uniquely extends to a unitary operator $W_0 : \mathcal{H}^0 \to \mathcal{H}^0_\varphi$.

Proposition

Define

$$W = \begin{pmatrix} W_0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{L}(\mathcal{H}, \mathcal{H}_\varphi) \quad \text{and} \quad \omega = \begin{pmatrix} k^\circ & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H}),$$

Then W is a unitary operator and intertwines the triples

$$(\mathcal{A}^\circ_\theta, \mathcal{H}, \omega D \omega)_\sigma \qquad \text{and} \qquad (\mathcal{A}^\circ_\theta, \mathcal{H}_\varphi, D_\varphi)_\sigma.$$

Corollary

$(\mathcal{A}^\circ_\theta, \mathcal{H}_\varphi, D_\varphi)_\sigma$ *is a twisted spectral triple.*

Gauss-Bonnet Theorem for NC Tori

Theorem (Gauss-Bonnet Theorem)

Let (Σ, g) be a compact Riemann surface. Define

$$\zeta\Delta_g; 0) = \lim_{s \to 0} \operatorname{Tr} \Delta_g^{-s}.$$

Then

$$\zeta(\Delta_g; 0) + 1 = \frac{1}{12\pi} \int_M \kappa(x) \sqrt{g(x)} dx = \frac{1}{16} \chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic and $\kappa(x)$ the scalar curvature. In particular $\zeta(\Delta_g; 0)$ is a topological invariant and a conformal invariant.

Theorem (Connes-Tretkoff)

Set $\Delta_\varphi = \partial_\varphi^* \partial$. Then the value of $\zeta(\Delta_\varphi; 0)$ is independent of the choice of the conformal weight φ, and hence is a conformal invariant.

Gauss-Bonnet Theorem for NC Tori

Conjecture (Gihyun Lee + Hyun-su Ha + RP)

Let (g_{ij}) be a Riemannian metric on $\mathcal{A}_\theta$ (i.e., a positive element of $M_2(\mathcal{A}_\theta)$) and Δ_g the associated Laplacian. Then the value of $\zeta(\Delta_g; 0)$ is independent of the choice of g, and hence is a topological invariant.