Introduction to Noncommutative Geometry
Part 1: Quantized Calculus

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Plan of Lectures

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- Part 1: *Quantized Calculus (aka Noncommutative Differentiable Calculus).*
- Part 2: *Spectral Triples (aka Noncommutative Manifolds). Examples.*
- Part 3: *Cyclic Cohomology (aka Noncommutative De Rham Theory). Examples.*
- Part 4: *The Local Index Formula in Noncommutative Geometry (Connes-Chern Character).*

References

Main References

- Alain Connes: *Noncommutative Geometry*.
- Joseph Varilly *et al.*: *Elements of Noncommutative Geometry*.
- University of Tokyo lecture notes (available on my website).

Remark

Slides of the lectures will be posted on my website.

Quantum Mechanics vs. General Relativity

Fundamental Problems

- Unify Gravity and Quantum Mechanics.
- Find a a common mathematical framework for general relativity and quantum mechanics.

NCG Approach

Translate the main tools of Riemannian geometry into the Hilbert space formalism of quantum mechanics.

Notation

In all what follows:

- $\mathcal{H}$ is a separable Hilbert space.
- $\mathcal{L}(\mathcal{H})$ is the C^*-algebra of bounded operators on $\mathcal{H}$.

NC Topological Spaces

Theorem (Gel'fand-Naimark)

Any C^*-algebra can be realized as a closed self-adjoint subalgebra of some $\mathcal{L}(\mathcal{H})$.

Theorem (Gel'fand-Naimark)

There is a one-to-one correspondence,

$$\begin{array}{ccc} \{\textit{Locally Compact Spaces}\} & \longleftrightarrow & \{\textit{Commutative } C^*\textit{-algebras}\} \\ X & \longrightarrow & C_0(X) \end{array}.$$

NCG Approach

- Extend Gel'fand duality to the setting of differential geometry.
- In this approach spaces are traded for algebras.

Example. Group Actions on Manifolds

Fact

Let G be a group acting by diffeomorphisms on a manifold M.

1. *If the action is free and proper, then M/G is a manifold.*
2. *In general, M/G need not be Hausdorff.*

NCG Approach

Trade the badly behaved space M/G for the crossed-product algebra $C_c^\infty(M) \rtimes G$.

Definition

$C_c^\infty(M) \rtimes G$ is $C_c^\infty(M) \otimes \mathbb{C}[G]$ equipped with

$$(f_1 \otimes \phi_1)(f_2 \otimes \phi_2) = f_1(f_2 \circ \phi_1^{-1}) \otimes \phi_1\phi_2,$$
$$(f \otimes \phi)^* = \overline{f} \otimes \phi^{-1}.$$

Theorem (Green)

If G acts freely and properly, then $C_c^\infty(M/G) \simeq C_c^\infty(M) \rtimes G$.

Quantized Calculus

Classical Calculus	**Quantized Calculus**
Complex Variable	Operator on $\mathcal{H}$
Real Variable	Selfadjoint Operator on $\mathcal{H}$
Infinitesimal Variable	???
Infinitesimal of Order α	???
Integral $\int f$	???

Infinitesimal Operators

Intuitive Definition

An infinitesimal is an object that it is smaller than any positive number.

Remark

For an operator $T \in \mathcal{L}(\mathcal{H})$ the condition

$$\|T\| < \epsilon \quad \text{for all } \epsilon > 0$$

gives the solution $T = 0$!

Definition (Infinitesimal Operator)

An operator $T \in \mathcal{L}(\mathcal{H})$ is infinitesimal when, for all $\epsilon > 0$, there is a subspace $E \subset \mathcal{H}$, $\dim E < \infty$, such that

$$\left\| T_{|E^\perp} \right\| < \epsilon.$$

Characteristic Values (aka Singular Values)

Definition (Characteristic Values)

The $(k+1)$-th characteristic value of $T \in \mathcal{L}(\mathcal{H})$ is

$$\mu_k(T) = \inf \left\{ \left\| T_{|E^\perp} \right\| ; \ \dim E = k \right\}.$$

Proposition (Min-Max Principle)

1. *We have*

$$\mu_k(T) = \inf \left\{ \| T - R \| ; \ \operatorname{rank} R \leq k \right\}.$$

2. *If T is compact, then*

$$\mu_k(T) = \ (k+1)\text{-th eigenvalues of } |T| = \sqrt{T^* T}.$$

Characterization of Infinitesimal Operators

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. Then TFAE

1. *T is an infinitesimal operator.*
2. *$\mu_k(T) \to 0$ as $k \to \infty$.*
3. *T is the norm-limit of finite rank operators.*
4. *T is a compact operator.*

Properties of Characteristic Values

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$*. Then*

$$\mu_k(T) = \mu_k(T^*) = \mu_k(|T|),$$

$$\mu_k(ATB) \le \|A\|\mu_k(T)\|B\| \quad \forall A, B \in \mathcal{L}(\mathcal{H}),$$

$$\mu_k(U^*TU) = \mu_k(T) \quad \text{for any unitary } U \in \mathcal{L}(\mathcal{H}).$$

Proposition

Let $S, T \in \mathcal{L}(\mathcal{H})$*. Then*

$$\mu_{k+l}(S + T) \le \mu_k(S) + \mu_l(T),$$

$$\mu_{k+l}(ST) \le \mu_k(S)\mu_l(T).$$

Infinitesimal Operators of Order $\alpha > 0$

Definition

A (compact) operator T is an infinitesimal of order α, $\alpha > 0$, when

$$\mu_k(T) = \mathrm{O}(k^{-\alpha}) \quad \text{as } k \to \infty.$$

Proposition

For $j = 1, 2$ let T_j be an infinitesimal operator of order α_j. Then

1. *$T_1 + T_2$ is an infinitesimal of order $\min(\alpha_1, \alpha_2)$.*
2. *$T_1 T_2$ is an infinitesimal of order $\alpha_1 + \alpha_2$.*

The Banach Ideal $\mathcal{L}^{1+}$

Definition

Define

$$\mathcal{L}^{1+} = \left\{ T \in \mathcal{L}(\mathcal{H});\ \mu_k(T) = \mathrm{O}(k^{-1}) \right\},$$
$$\|T\|_{1+} = \sup_{k \geq 0} \{k\mu_k(T)\}, \quad T \in \mathcal{L}^{1+}.$$

Proposition

1. *$\mathcal{L}^{1+}$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$.*
2. *$\|\cdot\|_{1+}$ is a Banach norm on $\mathcal{L}^{1+}$ in such a way that*
$$\|ATB\|_{1+} \leq \|A\|\|T\|_{1+}\|B\| \qquad \forall A, B \in \mathcal{L}(\mathcal{H}).$$
3. *The closure in $\mathcal{L}^{1+}$ of the space of finite-rank operators is*
$$\mathcal{L}_0^{1+} = \left\{ T \in \mathcal{L}^{1+};\ \mu_k(T) = \mathrm{o}(k^{-1}) \right\}.$$
In particular, it contains all infinitesimals of order > 1.

Noncommutative Integral

Ansatz for a NC Integral

The NC integral should be a linear functional $⨍$ such that

1. It is defined on infinitesimals of order 1 (i.e., its domain contains $\mathcal{L}^{1+}$).
2. It vanishes on infinitesimals of order > 1.
3. It takes on non-negative values on positive operators.
4. It is unitary invariant, i.e., $⨍ U^* TU = ⨍ T$ for any unitary U.

Remark

- The last two conditions imply that $⨍$ must be a positive trace.
- The standard trace $T \to \text{Trace}(T)$ on trace-class operators is a positive linear trace. However, it satisfies none of the first two conditions:
  - An infitesimal of order 1 need not be trace-class.
  - The above trace is not annihilated by rank 1 projections.

The Dixmier Trace

Theorem (Dixmier '66)

There is a continuous positive linear trace on $\mathcal{L}^{1+}$ which is annihilated by operators in $\mathcal{L}^{1+}_0$ (including infinitesimals of order > 1).

Strategy

1. Construct a (positive) trace on $\mathcal{L}^{1+}_+ = \{T \in \mathcal{L}^{1+};\ T \geq 0\}$ which is continuous and annihilated by finite rank operators.
2. Extend it by linearity to all $\mathcal{L}^{1+}$

Limit Processes

Notation

- $\ell^\infty = \{(a_n)_{n\geq 0} \subset \mathbb{C};\ \sup|a_n|\}$.
- $\ell_0 = \{(a_n)_{n\geq 0} \subset \mathbb{C};\ \lim a_n| = 0\}$.

Facts

1. *ℓ^∞ is a commutative unital C^*-algebra.*
2. *ℓ_0 is a closed ideal of ℓ^∞.*
3. *Both ℓ^∞ and ℓ^∞/ℓ_0 are non-separable.*

Definition

A state on ℓ^∞/ℓ_0 is a linear functional $\omega : \ell^\infty/\ell_0 \to \mathbb{C}$ such that

1. $\omega(1) = 1$.
2. $\omega(x^*x) \geq 0$ for all $x \in \ell^\infty/\ell_0$.

Limit Processes

Definition

Let ω be a state on ℓ^∞/ℓ_0. Define $\lim_\omega : \ell^\infty \to \mathbb{C}$ by

$$\lim_\omega a_n = \omega([a_n]) \qquad \forall (a_n) \in \ell^\infty,$$

where $[a_n]$ is the class of (a_n) in ℓ^∞/ℓ_0.

Proposition

1. $\lim_\omega$ *is a positive linear functional on* ℓ^∞.
2. *If* $a_n \geq 0$ *for all* n, *then*

$$\lim_{n\to\infty} a_n = L \iff \lim_\omega a_n = L \quad \forall \omega.$$

Remark

The last part follows from the fact that the states separate the points (any character is a state).

Partial Traces $\sigma_N(T)$

Notation

For $T \in \mathcal{L}(\mathcal{H})$ set
$$\sigma_N(T) = \sum_{k<N} \mu_k(T).$$

Remarks

1. We have
$$\text{Trace}(|T|) = \sum_{k=0}^{\infty} \mu_k(T) = \lim_{N\to\infty} \sigma_N(T).$$
2. If $T \in \mathcal{L}_+^{1+}$, then
$$\sigma_N(T) = \mathrm{O}(\log N), \quad \text{i.e., } (\log N)^{-1}\sigma_N(T) \in \ell^\infty.$$
3. If $T \in \mathcal{L}_0^{1+}$, then
$$\sigma_N(T) = \mathrm{o}(\log N), \quad \text{i.e., } (\log N)^{-1}\sigma_N(T) \in \ell_0.$$

The Dixmier Trace on $\mathcal{L}^{1+}$

Lemma

Let T_1 and T_2 be in $\mathcal{L}^{1+}_+$. Then

$$\sigma_N(T_1 + T_2) = \sigma_N(T_1) + \sigma_N(T_2) + O(1).$$

Thus,

$$\frac{\sigma_N(T_1 + T_2)}{\log N} = \frac{\sigma_N(T_1)}{\log N} + \frac{\sigma_N(T_2)}{\log N} \mod \ell_0.$$

Definition

Let ω be a state on ℓ^∞/ℓ_0. Define $\mathrm{Tr}_\omega : \mathcal{L}^{1+}_+ \to [0, \infty)$ by

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{\sigma_N(T)}{\log N}.$$

Proposition

1. $\mathrm{Tr}_\omega(T_1 + T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2)$.
2. $\mathrm{Tr}_\omega(U^* TU) = \mathrm{Tr}_\omega(T)$ *for any unitary* $U \in \mathcal{L}(\mathcal{H})$.
3. *We have*

$$\lim_{N\to\infty} (\log N)^{-1}\sigma_N(T) = L \iff \mathrm{Tr}_\omega(T) = L \ \forall\omega.$$

4. *If* $T \in \mathcal{L}_0^{1+}$, *then* $\mathrm{Tr}_\omega(T) = 0$.

Proposition

The functional Tr_ω *has a unique linear extension* $\mathrm{Tr}_\omega : \mathcal{L}^{1+} \to \mathbb{C}$ *in such a way that*

1. *It is continuous positive trace on* $\mathcal{L}^{1+}$.
2. *It is annihilated by operators in* $\mathcal{L}_0^{1+}$ *(including infinitesimals of order* > 1*).*

The Dixmier Trace. Measurable Operators

Definition

The trace $\mathrm{Tr}_\omega : \mathcal{L}^{1+} \to \mathbb{C}$ is called the Dixmier trace associated with ω.

Definition

1. An operator $T \in \mathcal{L}^{1+}$ is measurable when the value of $\mathrm{Tr}_\omega(T)$ is independent of ω.
2. For such an operator we define

$$⨍ T := \mathrm{Tr}_\omega(T),$$

where ω is any state on ℓ^∞/ℓ_0.

Remark

The functional $⨍$ on measurable operators is also called the Dixmier trace.

Quantized Calculus

Classical Calculus	**Quantized Calculus**
Complex Variable	Operator on Hilbert space $\mathcal{H}$
Real Variable	Selfadjoint Operator on $\mathcal{H}$
Infinitesimal Variable	Compact Operator
Infinitesimal of Order α	Compact Operator T such that $\mu_n(T) = O(n^{-\alpha})$
Integral $\int f$	Dixmier Trace $⨍ T$

Example. Riemannian Manifolds

Setup/Notation

- $\mathcal{H} = L^2_g(M)$, where (M^n, g) is a compact Riemannian mfld.
- $\Delta_g = d^*d$ is the Laplace operator on functions.
- $\lambda_k(\Delta_g)$, $k \geq 0$, are the eigenvalues of Δ_g such that

$$0 = \lambda_0(\Delta_g) < \lambda_1(\Delta_g) \leq \lambda_2(\Delta_g) \leq \cdots .$$

Theorem (Weyl's Law)

As $k \to \infty$,

$$\lambda_k(\Delta_g) \sim \left(\frac{k}{c_n \operatorname{Vol}_g(M)} \right)^{\frac{2}{n}}, \qquad c_n = (4\pi)^{-\frac{n}{2}} \Gamma\left(\frac{n}{2} + 1\right)^{-1}.$$

Corollary

The operator $\Delta_g^{-\frac{n}{2}}$ is measurable and

$$⨍ \Delta_g^{-\frac{n}{2}} = c_n \operatorname{Vol}_g(M).$$

Pseudodifferential Operators (aka ΨDOs)

Main Properties of ΨDOs

1. There are continuous linear operators $C^\infty(M) \to C^\infty(M)$.
2. They form a filtered algebra,
$$\Psi^\bullet(M) = \bigcup_{m\in\mathbb{Z}} \Psi^m,$$
$$P_1 \in \Psi^{m_1}(M) \text{ and } P_2 \in \Psi^{m_2}(M) \implies P_1P_2 \in \Psi^{m_1+m_2}(M).$$
3. If $P \in \Psi^m(M)$ with $m \leq 0$, then P extends to a bounded operator,
$$P : L^2(M) \longrightarrow L^2(M).$$
This operator is compact (resp., trace-class) when $m < 0$ (resp., $m < -n$).

Examples

1. A differential operator of order m is a ΨDO of order m.
2. If P is an elliptic operator of order m, then P^{-1} is a ΨDO of order $-m$.
3. $\Delta_g^{\frac{m}{2}}$, $m \in \mathbb{Z}$, is a ΨDO of order m.

Noncommutative Residue

Theorem (Guillemin, Wodzicki)

1. *The following formula defines a linear form on $\Psi^\bullet(M)$.*

$$\operatorname{Res}(P) = \operatorname{Res}_{z=0} \operatorname{Trace}\left(P\Delta_g^{-\frac{z}{2}}\right) \quad \textit{for all } P \in \Psi^\bullet(M).$$

   It is called the noncommutative residue.

2. *We have*

$$\operatorname{Res}(P) = \int_M c_P(x)\sqrt{g(x)}d^n x,$$

   where $c_P(x)$ is computable in terms of the "symbol" of P.

3. *The NC residue is a trace, i.e., $\operatorname{Res}[P_1 P_2] = \operatorname{Res}[P_2 P_1]$.*

4. *The NC residue vanishes on ΨDOs of order $< -n$.*

Theorem (Wodzicki)

If M is connected, then every trace on $\Psi^\bullet(M)$ is a scalar multiple of the noncommutative residue.

Theorem (Connes)

Let $P \in \Psi^{-m}(M)$, $m > 0$.

1. *P is an infinitesimal of order m/n.*
2. *If $m = n$, then P is measurable and*

$$\fint P = \frac{1}{n}\mathrm{Res}(P).$$

Consequence

We can integrate any ΨDO even if it is not an infinitesimal of order 1 (or is even bounded) by letting

$$\fint P := \frac{1}{n}\mathrm{Res}(P) \qquad \text{for all } P \in \Psi^{\bullet}(M).$$

Remark

Connes' theorem is true for any trace on $\mathcal{L}^{1+}$ (Kalton *et al.*).

Applications

Proposition

For all $f \in C^\infty(M)$, *we have*

$$\fint f \Delta_g^{-\frac{n}{2}} = c_n \int_M f(x)\sqrt{g(x)}d^n x.$$

Consequence

- The operator $\Delta_g^{-\frac{n}{2}}$ recaptures the volume element $\sqrt{g(x)}d^n x$.
- This enables us to define volumes of any dimension (e.g., length, area) of a Riemannian manifold (*cf.* last three slides).

Applications

Proposition

For all $f \in C^\infty(M)$, we have

$$\fint f \Delta_g^{-\frac{n-2}{2}} = c'_n \int_M f(x)\kappa_g(x)\sqrt{g(x)}d^n x, \quad c'_n = \frac{(4\pi)^{-\frac{n}{2}}}{3n\Gamma\left(\frac{n-2}{2}\right)}$$

where $\kappa_g(x)$ is the scalar curvature.

Consequences

- The operator $(c'_n)^{-1}\Delta_g^{-\frac{n-2}{2}}$ recaptures the "dressed" scalar curvature $\sqrt{g(x)}\kappa_g(x)$.
- For $f = 1$ we recover the Einstein-Hilbert action from Gravity:

$$\mathcal{I}_{\mathrm{EH}} := \int_M \kappa_g(x)\sqrt{g(x)}d^n x = (c'_n)^{-1} \fint \Delta_g^{-\frac{n-2}{2}}.$$

This provides us with a spectral theoretic interpretation of $\mathcal{I}_{\mathrm{EH}}$.

Noncommutative Length Element

Observations

1. $c_n^{-1}\Delta_g^{-\frac{n}{2}}$ recaptures the volume element.
2. Intituively, the volume element is $(ds)^n$, where ds is the length element.
3. That is, ds is an n-th root of the volume element.

Definition (Length Element)

The (noncommutative) length element is

$$ds := \left(c_n^{-1}\Delta_g^{-\frac{n}{2}}\right)^{\frac{1}{n}} = (c_n)^{-\frac{1}{n}}\Delta_g^{-\frac{1}{2}}.$$

Lower Dimensional Volumes

Observations

1. The length (resp., area) is the "integral" of ds (resp., ds^2).
2. The volume of dimension k is the "integral" of ds^k.

Definition

For $k = 1, \ldots, n$, the volume of dimension k of (M^n, g) is

$$\mathrm{Vol}_g^{(k)}(M) := \int\!\!\!\!\!- \, ds^k = (c_n)^{\frac{k}{n}} \int\!\!\!\!\!- \, \Delta_g^{-\frac{k}{2}}.$$

Remarks

1. For $k = n$ we recover the standard volume of (M^n, g).
2. For $k = 1$ (resp., $k = 2$) this is interpreted as the length (resp., area) of (M, g).

Lower Dimensional Volumes

Proposition

For $k = 1, \dots, n$, we have

$$\mathrm{Vol}_g^{(k)}(M) = \int_M I_g^{(k)}(x)\sqrt{g(x)}d^n x,$$

where $I_g^{(k)}$ is a universal polynomial in the inverse metric g^{ij} and the covariant derivatives of the curvature tensor R_{ijkl}.

Remark

For the previous discussions it is more natural to replace $\Delta_g^{\frac{1}{2}}$ by the Dirac operator.